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Lie group action and stability analysis of stationary solutions for a free boundary problem modelling tumor growth

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ABSTRACT

In this paper we study asymptotic behavior of solutions for a free boundary problem modelling tumor growth. We first establish a general result for differential equations in Banach spaces possessing a Lie group action which maps a solution into new solutions. We prove that a center manifold exists under certain assumptions on the spectrum of the linearized operator without assuming that the space in which the equation is defined is of either $D_A(\theta)$ or $D_A(\theta,\infty)$ type. By using this general result and making delicate analysis of the spectrum of the linearization of the stationary free boundary problem, we prove that if the surface tension coefficient γ is larger than a threshold value γ^* then the unique stationary solution is asymptotically stable modulo translations, provided the constant c is sufficiently small, whereas if $\gamma<\gamma^*$ then this stationary solution is unstable.

1. Introduction

This paper aims at studying asymptotic behavior of solutions of the following free boundary problem:

$$c\partial_t\sigma=\Delta\sigma-f(\sigma),\quad x\in\Omega(t),\ t>0, \tag{1.1}$$

$$-\Delta p=g(\sigma),\quad x\in\Omega(t),\ t>0, \tag{1.2}$$

$$\sigma=\bar\sigma,\quad x\in\partial\Omega(t),\ t>0, \tag{1.3}$$

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$$p = \gamma\kappa, \quad x \in \partial\Omega(t),\ t > 0, \tag{1.4}$$

$$\mathbf{V} = -\partial_{\mathbf{n}} p, \quad x \in \partial\Omega(t),\ t > 0, \tag{1.5}$$

$$\sigma(x, 0) = \sigma_0(x), \quad x \in \Omega_0, \tag{1.6}$$

$$\Omega(0) = \Omega_0. \tag{1.7}$$

Here $\sigma = \sigma(x,t)$ and $p = p(x,t)$ are unknown functions defined on the space–time manifold $\bigcup_{t\geqslant 0}(\overline{\Omega(t)} \times \{t\})$, and $\Omega(t)$ is an a priori unknown bounded time-dependent domain in $\mathbb{R}^n$, whose boundary $\partial\Omega(t)$ has to be determined together with the unknown functions σ and p. Besides, f and g are given functions, c, $\bar{\sigma}$ are γ are positive constants, κ, $\mathbf{V}$ and $\mathbf{n}$ are the mean curvature, the normal velocity and the unit outward normal vector of $\partial\Omega(t)$, respectively, and σ_0, Ω_0 are given initial data of $\sigma = \sigma(\cdot, t)$ and $\Omega = \Omega(t)$, respectively. The sign of κ is fixed on by the condition that $\kappa \geqslant 0$ at points where $\partial\Omega(t)$ is convex with regard to $\Omega(t)$.

The above problem arises from recently developed subject of tumor growth modelling. It models the growth of tumors cultivated in laboratory or so-called *multicellular spheroids* [1,6,7,26,28,30,34]. In this model $\Omega(t)$ represents the domain occupied by the tumor at time t, σ and p stand for the nutrient concentration and the tumor tissue pressure, respectively, and $f(\sigma)$, $g(\sigma)$ are the nutrient consumption rate and the tumor cell proliferation rate, respectively. It is assumed that all tumor cells are alive and dividable, and their density is constant, so that in f and g no cell density argument is involved. It is also assumed that the tumor is cultivated in a solution of nutrition materials whose concentration keeps constant in the process of cultivation, and $\bar{\sigma}$ reflects this constant nutrient supply to the tumor. The term $\gamma\kappa$ on the right-hand side of (1.4) stands for surface tension of the tumor. Eq. (1.5) reflects the fact that the normal velocity of the tumor surface is equal to the normal component of the movement velocity of tumor cells adjacent to the surface. For more details of the modelling we refer the reader to see Refs. [1,6,7,9,11,14–16] and [26]. Here we point out that, by rescaling which we have pre-assumed and did not particularly mention, the constant c represents the ratio between the nutrient diffusion time and the tumor-cell doubling time, so that $c \ll 1$, cf. [1,6] and [7]. Finally, we make the following assumptions on the functions f and g:

$(A1)$ $f \in C^\infty[0, \infty)$, $f'(\sigma) > 0$ for $\sigma \geqslant 0$ and $f(0) = 0$.
$(A2)$ $g \in C^\infty[0, \infty)$, $g'(\sigma) > 0$ for $\sigma \geqslant 0$ and there exists a number $\tilde{\sigma} > 0$ such that $g(\tilde{\sigma}) = 0$ ($\Rightarrow g(\sigma) < 0$ for $0 \leqslant \sigma < \tilde{\sigma}$ and $g(\sigma) > 0$ for $\sigma > \tilde{\sigma}$).
$(A3)$ $\tilde{\sigma} < \bar{\sigma}$.

These assumptions are based on biological considerations, see [11,15] and [16].

Local well-posedness of the above problem has been recently established by the author in a more general framework in Ref. [14] by using the analytic semigroup theory, which extends and modifies an earlier work of Escher [20] for the special case that $f(\sigma) = f(\sigma)$ but $g(\sigma) = \mu(\sigma - \tilde{\sigma})$. In this paper we consider the more difficult topic of asymptotic behavior of the solution. More precisely, from [11] and [15] we know that under the above assumptions $(A1)$–$(A3)$, the system (1.1)–(1.5) has a radially symmetric stationary solution $(\sigma_s, p_s, \Omega_s)$, which is unique up to translations and rotations of the coordinate of $\mathbb{R}^n$ and globally asymptotically stable under radially symmetric perturbations. This paper aims at studying the following question: Is $(\sigma_s, p_s, \Omega_s)$ also asymptotically stable under non-symmetric perturbations?

We first make a short review to previous work on this topic. Rigorous analysis of free boundary problems of partial differential equations arising from tumor growth modelling has attracted a lot of attention during the past several years, and many interesting results have been systematically derived, cf. [3,4,8–18,20,22–25], and the references cited therein. As far as the problem (1.1)–(1.7) and its certain more specific forms are concerned, we cite Refs. [3,4,9,11,14–16,20,22–24]. In particular, in [23] Friedman and Reitich considered radially symmetric version of the problem (1.1)–(1.7) in the special case that $f(\sigma) = \lambda\sigma$ and $g(\sigma) = \mu(\sigma - \tilde{\sigma})$. Under the assumption $(A3)$, they proved the following results: (1) The problem is globally well-posed. (2) There exists a unique stationary solution. (3) For c sufficiently small this stationary solution is globally asymptotically stable. (4) For c large the stationary solution is unstable. The author of the present paper has recently extended the assertions (1), (2), (3) to the general case that f and g are general functions satisfying the conditions $(A1)$–$(A3)$, see [11]. The general non-symmetric version of (1.1)–(1.7) in the special case that $f(\sigma) = \lambda\sigma$ and $g(\sigma) = \mu(\sigma - \tilde{\sigma})$ has also been systematically studied by Friedman and his collaborators. Bazaliy and Friedman investigated local well-posedness of the time-dependent problem in Ref. [3]. In [4] they studied asymptotic behavior of the solution starting from a neighborhood of the unique radially symmetric stationary solution ensured by the above assertion (2), and proved that, for $c = 1$, $\lambda = 1$, $\gamma = 1$ and μ sufficiently small, the radially symmetric stationary solution is (locally) asymptotically stable under non-radial perturbations. This work was recently refined by Friedman and Hu [22]. They proved that, again for $c = 1$, $\lambda = 1$ and $\gamma = 1$, there exists a threshold value $\mu^* > 0$, such that for $0 < \mu < \mu^*$ the radially symmetric stationary solution is (locally) asymptotically stable under non-radial perturbations, while for $\mu > \mu^*$ this stationary solution is unstable.

In a recent work of the present author jointly with Escher [16], the problem (1.1)–(1.7) with general functions f and g satisfying $(A1)$–$(A3)$ but $c = 0$ was studied. We proved that there exists a threshold value $\gamma_* > 0$, the supremum of all bifurcation points γ_k ($k = 2, 3, \ldots$, see [15]), such that if $\gamma > \gamma_*$ then the radially symmetric stationary solution $(\sigma_s, p_s, \Omega_s)$ is (locally) asymptotically stable *modulo translations*, i.e., any solution starting from a small neighborhood of $(\sigma_s, p_s, \Omega_s)$ is global and, as $t \to \infty$, it converges to either $(\sigma_s, p_s, \Omega_s)$ or an adjacent stationary solution $(\sigma_s', p_s', \Omega_s')$ obtained by translating $(\sigma_s, p_s, \Omega_s)$ (recall that any translation of $(\sigma_s, p_s, \Omega_s)$ is still a stationary solution), whereas if $\gamma < \gamma_*$ then $(\sigma_s, p_s, \Omega_s)$ is unstable.

In this paper we want to extend the above result of [16] for the degenerate case $c = 0$ to the more difficult non-degenerate case $c \neq 0$, assuming that c is sufficiently small. The main idea of analysis is the same with that of [16], namely, we shall first reduce the PDE problem into a differential equation in a Banach space and next use the abstract geometric theory for parabolic differential equations in Banach spaces to get the desired result. However, unlike in [16] where we used the well-developed center manifold theorem by Da Prato and Lunardi [19] to make the analysis, in this paper we shall have to first establish a new center manifold theorem, because the above-mentioned center manifold of Da Prato and Lunardi is not applicable to the case $c \neq 0$. The reason is as follows. Recall that the center manifold theorem of Da Prato and Lunardi requires the Banach space in which the differential equation is considered must be of the type either $D_A(\theta)$, the continuous interpolation space, or $D_A(\theta, \infty)$, the real interpolation space of the type (θ, ∞) $(0 < \theta < 1)$. Such spaces cannot be reflexive (cf. [2,33]). In the degenerate case $c = 0$ the reduced equation contains only the unknown function ρ defining the free boundary $\partial\Omega(t)$, which is a quasi-linear parabolic pseudo-differential equation on a compact manifold, so that no boundary conditions appear and we can thus work on the little Hölder space $h^{m+\alpha}$ which is of the type $D_A(\theta)$. In the present non-degenerate case $c \neq 0$, however, since the reduced equation contains not only ρ but also the unknown σ, the Dirichlet boundary condition for σ renders it impossible for us to work on a space of the type either $D_A(\theta)$ or $D_A(\theta, \infty)$.

To remedy this deficiency, in this paper we shall first establish a new center manifold theorem which removes this very restrictive assumption on the space X, but instead we shall assume that the equation admits a local Lie group action by which a solution is mapped into new solutions. We shall show that the phase diagram of a differential equation possessing such a Lie group action has a very nice structure and its center manifold can be very easily obtained. In particular, this new center manifold theorem does not make any additional assumption on the structure of the space X. Since the differential equation reduced from the problem (1.1)–(1.7) naturally possesses a Lie group action induced by translations of the coordinate of $\mathbb{R}^n$, by using this new center manifold result we are able to make analysis in the framework of Sobolev and Besov spaces. Our final result says that similar assertions as for the case $c = 0$ also hold for the case that c is non-vanishing but very small, and this result will be established in the space $W^{m-1,q} \times W^{m-3,q} \times B_{qq}^{m-1/q}$, where $W^{m-1,q}$ and $B_{qq}^{m-1/q}$ represent the Sobolev and Besov spaces, respectively.

It should be noted that our center manifold theorem for differential equations in Banach spaces possessing Lie group action established in this paper not only works for the tumor model (1.1)–(1.7) as well as its special form of the case $c = 0$, but also applies to other problems such as the Hele-Shaw problem. Thus, the center manifold theorem established in this paper has its own theoretic importance. More applications of this result will be given in our future work.

To give a precise statement of our main result, let us first introduce some notation. Recall that the radially symmetric stationary solution $(\sigma_s, p_s, \Omega_s)$ of (1.1)–(1.5), where $\Omega_s = \{r < R_s\}$ with $r = |x|$, is the unique solution of the following free boundary problem:

$$\sigma_s''(r) + \frac{n-1}{r}\sigma_s'(r) = f\big(\sigma_s(r)\big), \quad 0 < r < R_s, \tag{1.8}$$

$$p_s''(r) + \frac{n-1}{r}p_s'(r) = -g\big(\sigma_s(r)\big), \quad 0 < r < R_s, \tag{1.9}$$

$$\sigma_s'(0) = 0, \qquad \sigma_s(R) = \bar{\sigma}, \tag{1.10}$$

$$p_s'(0) = 0, \qquad p_s(R_s) = \frac{\gamma}{R_s}, \tag{1.11}$$

$$p_s'(R_s) = 0. \tag{1.12}$$

For $z \in \mathbb{R}^n$, we denote

$$\sigma_s^z(x) = \sigma_s\big(|x-z|\big), \qquad p_s^z(x) = p_s\big(|x-z|\big), \qquad \Omega_s^z = \big\{x \in \mathbb{R}^n\colon\ |x-z| < R_s\big\}.$$

Clearly, for any $z \in \mathbb{R}^n$ the triple $(\sigma_s^z, p_s^z, \Omega_s^z)$ is a stationary solution of the system (1.1)–(1.5). If $|z|$ is sufficiently small then there exists a unique $\rho_s^z \in C^\infty(\mathbb{S}^{n-1})$ which is sufficiently close to the constant function R_s, such that

$$\Omega_s^z = \big\{r < \rho_s^z(\omega),\ \omega \in \mathbb{S}^{n-1}\big\}.$$

Since we shall only consider solutions of (1.1)–(1.7) which are close to the stationary solution $(\sigma_s, p_s, \Omega_s)$, we can write $\Omega(t)$ as $\Omega(t) = \{r < \rho(\omega, t),\ \omega \in \mathbb{S}^{n-1}\}$ for some $\rho(\cdot, t) \in C(\mathbb{S}^{n-1})$ for every $t > 0$, and, correspondingly, we write Ω_0 as $\Omega_0 = \{r < \rho_0(\omega),\ \omega \in \mathbb{S}^{n-1}\}$, where $\rho_0 \in C(\mathbb{S}^{n-1})$. Finally, from [15] we know that the linearization of the stationary version of (1.1)–(1.5) has infinite many eigenvalues γ_k, $k = 2, 3, \ldots$, which are all positive and converge to zero as $k \to \infty$. As in [16] we set

$$\gamma_* = \max\{\gamma_k,\ k = 2, 3, \ldots\}.$$

The main result of this paper is as follows:

Theorem 1.1. *If $\gamma > \gamma_*$ then there exists a corresponding $c_0 > 0$ such that for any $0 < c < c_0$, the stationary solution $(\sigma_s, p_s, \Omega_s)$ of (1.1)–(1.5) is asymptotically stable modulo translations in the following sense: There exists $\varepsilon > 0$ such that for any $\rho_0 \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ and $\sigma_0 \in W^{m,q}(\Omega_0)$ ($m \in \mathbb{N}$, $m \geqslant 5$, $1 \leqslant q < \infty$ and $q > n/(m-4)$) satisfying*

$$\|\rho_0 - R_s\|_{B_{qq}^{m-1/q}(\mathbb{S}^{n-1})} < \varepsilon, \qquad \|\sigma_0 - \sigma_s\|_{W^{m,q}(\Omega_0)} < \varepsilon, \qquad \sigma_0|_{\partial\Omega_0} = \bar{\sigma},$$

the problem (1.1)–(1.7) has a unique solution (σ, p, Ω) (in the sense of Theorem 1.1 of [14]) for all $t \geqslant 0$, and there exists $z \in \mathbb{R}^n$ uniquely determined by ρ_0 and σ_0 such that

$$\big\|\sigma(\cdot,t) - \sigma_s^z\big\|_{W^{m-1,q}(\Omega(t))} + \big\|p(\cdot,t) - p_s^z\big\|_{W^{m-3,q}(\Omega(t))} + \big\|\rho(\cdot,t) - \rho_s^z\big\|_{B_{qq}^{m-1/q}(\mathbb{S}^{n-1})} \leqslant Ce^{-\kappa t}$$

for some $C > 0$, $\kappa > 0$ and all $t \geqslant 0$. If $\gamma < \gamma_*$ then there also exists a corresponding $c_0 > 0$ such that for any $0 < c < c_0$, $(\sigma_s, p_s, \Omega_s)$ is unstable.

Remark 1.1. By the assertion (4) of Friedman and Reitich reviewed before, we see that the condition $c < c_0$ cannot be removed. Besides, as we mentioned earlier, though we only consider solutions in $W^{m-1,q} \times W^{m-3,q} \times B_{qq}^{m-1/q}$, a similar result surely also holds for solutions in the space $C^{m+\alpha} \times C^{m-2+\alpha} \times C^{m+\alpha}$. In addition, the conditions $m \geqslant 5$ and $n/(m-4) < q < \infty$ can be weakened up to $m \geqslant 3$ and $n/(m-2) < q < \infty$. To achieve this improvement we need a modified version of Theorem 2.1 of the next section; see Remark 2.1 in the end of Section 2.

The proof of the above theorem will be given in the last section of this paper, after step-by-step preparations in Sections 2–6. The layout of the rest part is as follows. In Section 2 we establish the general result for differential equations in Banach spaces mentioned earlier. In Section 3 we first use the so-called Hanzawa transformation to transform the problem (1.1)–(1.7) into an equivalent problem on the fixed domain Ω_s, which for simplicity of notation will be assumed to be the unit sphere $\mathbb{B}^n$ later on, and next we further reduce the PDE problem into a differential equation in the Banach space $W^{m-3,q}(\mathbb{B}^n) \times B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$ for the unknowns (σ, ρ). In Section 4 we construct Lie group action for the reduced differential equation. In Section 5 we compute the linearization of the reduced equation. Section 6 aims at studying the spectrum of the linearized problem. In the last section we complete the proof of Theorem 1.1.

2. An abstract result

Let X and X_0 be two Banach spaces such that $X_0 \hookrightarrow X$. We particularly emphasize that X_0 need not be dense in X. Let $\mathcal{O}$ be an open subset of X_0. Let $F \in C^{2-0}(\mathcal{O}, X)$, i.e. $F \in C^1(\mathcal{O}, X)$ and F' $(= DF =$ the Fréchet derivative of $F)$ is Lipschitz continuous. In this section we consider the initial value problem

$$\begin{cases} u'(t) = F\big(u(t)\big), & t > 0, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $u_0 \in \mathcal{O}$. By a *solution* of (2.1) we mean a solution of the class $u \in C([0,T), X) \cap C((0,T), \mathcal{O}) \cap L^\infty((0,T), \mathcal{O}) \cap C^1((0,T), X)$ defined in a maximal existence interval $I = [0,T)$ $(0 < T \leqslant \infty)$, which satisfies (2.1) in $[0,T)$ in usual sense and is not extendable. If u satisfies the stronger condition $u \in C([0,T), \mathcal{O}) \cap C^1([0,T), X)$ then we call it a *strict solution*. Later on we shall denote by $u(t, u_0)$ the solution of (2.1) when it exists and is unique. We always assume that for some $u_s \in \mathcal{O}$ there holds $F(u_s) = 0$, so that $u(t) = u_s$, $t \geqslant 0$, is a stationary solution of the equation $u' = F(u)$. We want to study asymptotic stability of u_s. Our first assumption is as follows:

(B_1) $A = F'(u_s)$ is a sectorial operator in X with domain X_0, and the graph norm of A is equivalent to the norm of X_0: $\|u\|_{X_0} \sim \|u\|_X + \|Au\|_X$.

Next, we consider some invariance property of F. Let G be a *local Lie group of dimension n* in the sense of Pontryagin [32]. Let $\mathcal{O}'$ be an open subset of X such that $\mathcal{O} \subseteq \mathcal{O}'$. Let $\mathcal{O}_1$ be an open subset of X_0 contained in $\mathcal{O}$, and $\mathcal{O}_1'$ be an open subset of X contained in $\mathcal{O}'$, such that $u_s \in \mathcal{O}_1 \subseteq \mathcal{O}_1'$. We assume that there is a continuous mapping $p : G \times \mathcal{O}_1' \to \mathcal{O}'$, such that

(i) $p(G \times \mathcal{O}_1) \subseteq \mathcal{O}$, and $p : G \times \mathcal{O}_1 \to \mathcal{O}$ is continuous;
(ii) $p(e, u) = u$ for every $u \in \mathcal{O}_1'$, where e denotes the unit of G, and $p(\sigma, p(\tau, u)) = p(\sigma\tau, u)$ for any $u \in \mathcal{O}_1'$ and $\sigma, \tau \in G$ such that $\sigma\tau$ is well defined and $p(\tau, u) \in \mathcal{O}_1'$;
(iii) if $\sigma, \tau \in G$ such that $p(\sigma, u) = p(\tau, u)$ for some $u \in \mathcal{O}_1'$ then $\sigma = \tau$;
(iv) for any $\sigma \in G$, the mapping $u \to p(\sigma, u)$ from $\mathcal{O}_1'$ to $\mathcal{O}'$ is Fréchet differentiable at every point in $\mathcal{O}_1$, and $[u \to D_u p(\sigma, u)] \in C(\mathcal{O}_1, L(X))$. Here as usual $L(X)$ denotes the Banach algebra of all bounded linear operators from X to itself;
(v) for any $u \in \mathcal{O}_1$, the mapping $\sigma \to p(\sigma, u)$ from G to $\mathcal{O}$ is continuously Fréchet differentiable when regarded as a mapping from G to X $(\Rightarrow D_\sigma p(\sigma, u) \in L(T_\sigma(G), X)$, and $[\sigma \to p(\sigma, u)] \in C^1(G, X))$. Moreover, $\operatorname{rank} D_\sigma p(\sigma, u) = n$ for every $\sigma \in G$ and $u \in \mathcal{O}_1$.

Later on we denote $S_\sigma(u) = p(\sigma, u)$ for $\sigma \in G$ and $u \in \mathcal{O}_1$. Our second assumption is as follows:

(B_2) There is a local Lie group G satisfying the properties (i)–(v), such that for any $u \in \mathcal{O}_1$ and $\sigma \in G$ there holds

$$F\big(S_\sigma(u)\big) = DS_\sigma(u)F(u). \tag{2.2}$$

This assumption has some obvious inferences. First, it implies that for any $u_0 \in \mathcal{O}_1$ and $\sigma \in G$ there holds $u(t, S_\sigma(u_0)) = S_\sigma(u(t,u_0))$, namely, if $t \to u(t)$ is a solution of the equation $u' = F(u)$ with initial value u_0, then $t \to S_\sigma(u(t))$ is also a solution of this equation, with initial value $S_\sigma(u_0)$. In particular, for any $\sigma \in G$, $S_\sigma(u_s)$ is a stationary solution of $u' = F(u)$. Next, if u_s is more regular than (v) in the sense that $[\sigma \to p(\sigma, u_s)] \in C^1(G, X_0)$ (so that $D_\sigma p(\sigma, u_s) \in L(T_\sigma(G), X_0)$ for any $\sigma \in G$), then by differentiating the relation $F(S_\sigma(u_s)) = 0$ in σ at $\sigma = e$ we see that $DF(u_s)D_\sigma p(e, u_s)\xi = 0$ for any $\xi \in T_e(G)$, so that $A = DF(u_s)$ is degenerate, and $\dim \operatorname{Ker} A \geqslant n$. We now assume that

(B_3) $[\sigma \to p(\sigma, u_s)] \in C^1(G, X_0)$, $\dim \operatorname{Ker} A = n$, and the induced operator $\overline{A} : X_0/\operatorname{Ker} A \to X/\operatorname{Ker} A$ of A is an isomorphism.

Here and throughout this paper, by isomorphism from a Banach space X_1 to another Banach space X_2 we mean a linear mapping $T : X_1 \to X_2$ such that it is a 1–1 correspondence, and both T and T^{-1} are continuous (i.e., T is not merely a linear isomorphism, but a topological homeomorphism as well). Finally, we assume that

(B_4) $\omega_- \equiv -\sup\{\operatorname{Re}\lambda\colon\ \lambda \in \sigma(A) \setminus \{0\}\} = -\sup\{\operatorname{Re}\lambda\colon\ \lambda \in \sigma(\overline{A})\} > 0.$

We point out that the condition (B_3) is equivalent to the following condition:

(B_3') $\dim \operatorname{Ker} A = n$, $\operatorname{Range} A$ is closed in X, and $X = \operatorname{Ker} A \oplus \operatorname{Range} A$.

The proof of equivalence of (B_3) with (B_3') is simple, so that is omitted.

The main result of this section is as follows:

Theorem 2.1. *Let the assumptions (B_1)–(B_4) be satisfied. Then there exists a neighborhood $\mathcal{O}_2$ of u_s, $\mathcal{O}_2 \subseteq \mathcal{O}_1$, such that the following assertions hold*:

(1) *For any $u_0 \in \mathcal{O}_2$ the problem* (2.1) *has a unique solution $u(t, u_0)$ which exists for all $t \geqslant 0$, and if furthermore $F(u_0) \in \overline{X}_0$, then $u(t,u_0)$ is a strict solution.*

(2) *The center manifold of the equation $u' = F(u)$ in $\mathcal{O}_2$ is given by $\mathcal{M}^c = \{S_\sigma(u_s)\colon\ \sigma \in G\} \cap \mathcal{O}_2$, which is a C^{2-0} manifold of dimension n and consists of all stationary solutions of this equation in $\mathcal{O}_2$.*

(3) *There exists a C^{2-0} submanifold $\mathcal{M}^s \subseteq \mathcal{O}_2$ of codimension n in X_0 passing u_s, such that for any $u_0 \in \mathcal{M}^s$ there holds $\lim_{t\to\infty} u(t,u_0) = u_s$ and vice versa, i.e. $\mathcal{M}^s$ is the stable manifold of u_s in $\mathcal{O}_2$.*

(4) *For every $u_0 \in \mathcal{O}_2$ there exist a unique $\sigma \in G$ and a unique $v_0 \in \mathcal{M}^s$ such that $u_0 = S_\sigma(v_0)$, and we have*

$$\lim_{t\to\infty} u(t,u_0) = S_\sigma(u_s). \tag{2.3}$$

Moreover, for any $0 < \omega < \omega_-$ there exists corresponding $C = C(\omega) > 0$ such that

$$\big\|u(t,u_0) - S_\sigma(u_s)\big\|_{X_0} \leqslant Ce^{-\omega t}\big\|u_0 - S_\sigma(u_s)\big\|_{X_0} \quad \textit{for all } t \geqslant 0. \tag{2.4}$$

To prove this theorem, we need a preliminary lemma. Let X be a Banach space. Let $\alpha \in (0,1)$ and $T > 0$. Recall that (cf. the introduction of Chapter 4 of [29]) $C^\alpha_\alpha((0,T], X)$ is the Banach space of bounded mappings $u:(0,T]\to X$ such that $t^\alpha u(t)$ is uniformly α-Hölder continuous for $0<t\leqslant T$, with norm

$$\|u\|_{C^\alpha_\alpha((0,T],X)}=\sup_{0<t\leqslant T}\big\|u(t)\big\|_X+\sup_{0<s<t\leqslant T}\frac{\|t^\alpha u(t)-s^\alpha u(s)\|_X}{(t-s)^\alpha}.$$

Recall also that (cf. Section 4.4 of [29]) for $\omega>0$, $C^\alpha([T,\infty),X,-\omega)$ is the Banach space of mappings $u:[T,\infty)\to X$ such that $e^{\omega t}u(t)$ is bounded and uniformly α-Hölder continuous for $t\geqslant T$, with norm

$$\|u\|_{C^\alpha([T,\infty),X,-\omega)}=\sup_{t\geqslant T}\big\|e^{\omega t}u(t)\big\|_X+\sup_{t>s\geqslant T}\frac{\|e^{\omega t}u(t)-e^{\omega s}u(s)\|_X}{(t-s)^\alpha}.$$

Lemma 2.2. *Let X and X_0 be two Banach spaces such that $X_0\hookrightarrow X$. Let A be a sectorial operator in X with domain X_0. Assume that $\omega_-=-\sup\{\operatorname{Re}\lambda:\ \lambda\in\sigma(A)\}>0$ and $f\in C^\alpha_\alpha((0,1],X)\cap C^\alpha([1,\infty),X,-\omega)$, where $\alpha\in(0,1)$ and $\omega\in(0,\omega_-)$. Let $u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}f(s)\,ds$, where $u_0\in X_0$. Then $u\in C^\alpha_\alpha((0,1],X_0)\cap C^\alpha([1,\infty),X_0,-\omega)$, and there exists constant $C=C(\alpha,\omega)>0$ independent of f and u_0 such that*

$$\|u\|_{C^\alpha_\alpha((0,1],X_0)}+\|u\|_{C^\alpha([1,\infty),X_0,-\omega)}\leqslant C\big(\|u_0\|_{X_0}+\|f\|_{C^\alpha_\alpha((0,1],X)}+\|f\|_{C^\alpha([1,\infty),X,-\omega)}\big).\tag{2.5}$$

Proof. By Theorem 4.3.5 and Corollary 4.3.6(ii) of [29] we have $\|u\|_{C^\alpha_\alpha((0,1],X_0)}\leqslant C(\|u_0\|_{X_0}+\|f\|_{C^\alpha_\alpha((0,1],X)})$, and by Proposition 4.4.10(i) of [29] we have $\|u\|_{C^\alpha([1,\infty),X_0,-\omega)}\leqslant C(\|u_0\|_X+\|f\|_{L^1([0,1],X)}+\|f\|_{C^\alpha([\frac12,\infty),X,-\omega)})$. Hence (2.5) holds. $\square$

Proof of Theorem 2.1. Without loss of generality we assume that $u_s=0$. Since we are studying solutions of (2.1) in a neighborhood of 0, by the assumption (B_1) and a standard perturbation result, we may assume that $F'(u)$ is a sectorial operator for every $u\in\mathcal{O}$ (with domain X_0), and the graph norm of $F'(u)$ is equivalent to the norm of X_0. It follows by a standard result (cf. Theorem 8.1.1 of [29] and the remark in lines 8–12 on p. 341 of [29]) that for any $u_0\in\mathcal{O}$, the problem (2.1) has a unique local solution $u\in C([0,T],X)\cap C((0,T],\mathcal{O})\cap L^\infty((0,T),\mathcal{O})\cap C^1((0,T],X)\cap C^\alpha_\alpha((0,T],X_0)$, and if further $F(u_0)\in\overline{X}_0$ then $u\in C([0,T],\mathcal{O})\cap C^1([0,T],X)\cap C^\alpha_\alpha((0,T],X_0)$, where $T>0$ depends on u_0 and α is an arbitrary number in $(0,1)$. Moreover, denoting by $T^*(u_0)$ the supreme of all such T, we know that there exists a constant $\varepsilon>0$ independent of u_0 such that if $\|u(t,u_0)\|_{X_0}<\varepsilon$ for all $t\in[0,T^*(u_0))$, then $T^*(u_0)=\infty$ (cf. Proposition 9.1.1 of [29]).

Next we denote $\sigma_-(A)=\sigma(A)\setminus\{0\}$. Let Γ be a closed smooth curve in the complex plane which encloses 0 and separates it from $\sigma_-(A)$, and let P be the projection operator in X defined by

$$P=\frac{1}{2\pi i}\int_\Gamma R(\lambda,A)\,d\lambda.$$

Since $X=\operatorname{Ker}A\oplus\operatorname{Range}A$, we have $PX=PX_0=\operatorname{Ker}A$, $(I-P)X=\operatorname{Range}A$ (cf. Proposition A.2.2 of [29]), and $AP=0$. Let $A_-=(I-P)A|_{(I-P)X_0}:(I-P)X_0\to(I-P)X$. Then $\sigma(A_-)=\sigma(A)\setminus\{0\}$, so that $\sup\{\operatorname{Re}\lambda:\ \lambda\in\sigma(A_-)\}=-\omega_-<0$. Besides, by the assumption (B_3) we see that $A_-:(I-P)X_0\to(I-P)X$ is an isomorphism.

Let $\mathcal{M}^c=\{S_\sigma(0):\ \sigma\in G\}$. By (v) in the assumption (B_2) we see that $\mathcal{M}^c$ is a C^1 submanifold of X, and $\dim\mathcal{M}^c=n$. The equation $u=S_\sigma(0)$ $(\sigma\in G)$ gives a parametrization of $\mathcal{M}^c$ by G. We can also give a parametrization of $\mathcal{M}^c$ by PX as follows. For $u\in\mathcal{O}$ let $x=Pu$ and $y=(I-P)u$. Take two sufficiently small numbers $\delta>0$ and $\delta'>0$ such that $x\in B_1(0,\delta)$ and $y\in B_2(0,\delta')$ imply that $u=x+y\in\mathcal{O}_1$, where

$$B_1(0,\delta)=\big\{x\in PX:\ \|x\|_{X_0}<\delta\big\}\quad\text{and}\quad B_2(0,\delta')=\big\{y\in(I-P)X_0:\ \|y\|_{X_0}<\delta'\big\}.$$

For $(x, y) \in B_1(0,\delta) \times B_2(0,\delta')$ we denote $\mathcal{F}_1(x, y) = PF(x+y)$ and $\mathcal{F}_2(x, y) = (I-P)F(x+y)$. We have $\mathcal{F}_2 \in C^{2-0}(B_1(0,\delta) \times B_2(0,\delta'), (I-P)X)$, $\mathcal{F}_2(0,0) = 0$, and $D_y\mathcal{F}_2(0,0) = A_-$. Since $A_- : (I-P)X_0 \to (I-P)X$ is an isomorphism, by the implicit function theorem we infer that if δ is sufficiently small then there exists $\varphi \in C^{2-0}(B_1(0,\delta), B_2(0,\delta'))$ such that $\varphi(0) = 0$, $\mathcal{F}_2(x, \varphi(x)) = 0$ for every $x \in B_1(0,\delta)$, and for $(x, y) \in B_1(0,\delta) \times B_2(0,\delta')$, $\mathcal{F}_2(x, y) = 0$ if and only if $y = \varphi(x)$. It follows that the equation $\mathcal{F}_2(x, y) = 0$ defines a C^{2-0} submanifold $\mathcal{M}_0$ of dimension n. Since $F(S_\sigma(0)) = 0$ for every $\sigma \in G$, which particularly implies that $(I-P)F(S_\sigma(0)) = 0$ for every $\sigma \in G$, we conclude that $\mathcal{M}^c \cap B_1(0,\delta) \times B_2(0,\delta') = \mathcal{M}_0$. Hence, the equation $y = \varphi(x)$ gives a parametrization of $\mathcal{M}^c$ by PX. Furthermore, from this argument we also see that $\mathcal{F}_1(x, \varphi(x)) = PF(S_\sigma(0)) = 0$ for every $x \in B_1(0,\delta)$. Note that since $D_x\mathcal{F}_2(0,0) = (I-P)AP = 0$, we have $\varphi'(0) = -[D_y\mathcal{F}_2(0,0)]^{-1}D_x\mathcal{F}_2(0,0) = 0$.

Let $N(u) = F(u) - Au$ (for $u \in \mathcal{O}_1$), $\mathcal{N}_1(x, y) = PN(x+y)$ and $\mathcal{N}_2(x, y) = (I-P)N(x+y)$ (for $(x, y) \in B_1(0,\delta) \times B_2(0,\delta')$). Let $x_0 = Pu_0$ and $y_0 = (I-P)u_0$. Then (2.1) is equivalent to the following problem:

$$\begin{cases} x' = \mathcal{N}_1(x, y), \quad x(0) = x_0, \\ y' = A_- y + \mathcal{N}_2(x, y), \quad y(0) = y_0. \end{cases} \tag{2.6}$$

Let $(x, y) = (x(t), y(t))$ be the solution of (2.6) defined in a maximal interval $[0, T_*)$ such that it exists for all $t \in [0, T_*)$ and lies in $B_1(0,\delta) \times B_2(0,\delta')$. Since $(x, y) = (0, 0)$ is a solution defined for all $t \geqslant 0$, by continuous dependence of solutions on initial data, we see that there exists a neighborhood $\mathcal{O}_2$ of 0 contained in $B_1(0,\delta) \times B_2(0,\delta')$, such that for any $u_0 \in \mathcal{O}_2$ there holds $T_* > 1$. In the sequel we assume that $u_0 \in \mathcal{O}_2$ so that $T_* > 1$. Let $v(t) = y(t) - \varphi(x(t))$. Since $A_-\varphi(x) + \mathcal{N}_2(x, \varphi(x)) = \mathcal{F}_2(x, \varphi(x)) = 0$ and $\mathcal{N}_1(x, \varphi(x)) = \mathcal{F}_1(x, \varphi(x)) = 0$ for all $x \in B_1(0,\delta)$, we have

$$\begin{aligned} v'(t) &= A_- v(t) + \big[\mathcal{N}_2\big(x(t), y(t)\big) - \mathcal{N}_2\big(x(t), \varphi\big(x(t)\big)\big)\big] - \varphi'\big(x(t)\big)\big[\mathcal{N}_1\big(x(t), y(t)\big) - \mathcal{N}_1\big(x(t), \varphi\big(x(t)\big)\big)\big] \\ &\equiv A_- v(t) + \mathcal{G}(t), \end{aligned}$$

so that

$$v(t) = e^{tA_-}v(0) + \int_0^t e^{(t-s)A_-}\mathcal{G}(s)\,ds.$$

It follows by Lemma 2.2 that for any $0 < \alpha < 1$ and $\omega \in (0, \omega_-)$ we have

$$\|v\|_{C^\alpha_\alpha((0,1],X_0)} + \|v\|_{C^\alpha([1,T_*),X_0,-\omega)} \leqslant C\big(\big\|v(0)\big\|_{X_0} + \|\mathcal{G}\|_{C^\alpha_\alpha((0,1],X)} + \|\mathcal{G}\|_{C^\alpha([1,T_*),X,-\omega)}\big), \tag{2.7}$$

where $C^\alpha([1, T_*), X, -\omega)$ is defined similarly as $C^\alpha([1,\infty), X, -\omega)$, with ∞ replaced with T_*. Note that all assertions in Lemma 2.2 clearly hold when ∞ is replaced by any $T_* \in (1, \infty]$. By a similar argument as in the proof of Theorem 9.1.2 (more precisely, as in line 24, p. 342 through line 10, p. 343) of [29], we have

$$\|\mathcal{G}\|_{C^\alpha_\alpha((0,1],X)} \leqslant C\Big(\sup_{0<t\leqslant 1}\big\|u(t)\big\|_{X_0} + \sup_{0<t\leqslant 1}\big\|\tilde{u}(t)\big\|_{X_0}\Big)\|v\|_{C^\alpha_\alpha((0,1],X)} \tag{2.8}$$

and

$$\|\mathcal{G}\|_{C^\alpha([1,T_*),X,-\omega)} \leqslant C\Big(\sup_{0\leqslant t<T_*}\big\|u(t)\big\|_{X_0} + \sup_{0\leqslant t<T_*}\big\|\tilde{u}(t)\big\|_{X_0}\Big)\|v\|_{C^\alpha([1,T_*),X,-\omega)}, \tag{2.9}$$

where $\tilde{u}(t) = x(t) + \varphi(x(t))$, and C is a constant independent of T_*. Substituting (2.8), (2.9) into (2.7), we obtain

$$\|v\|_{C_\alpha^\alpha((0,1],X_0)} + \|v\|_{C^\alpha([1,T_*),X_0,-\omega)} \leqslant C\big[\big\|v(0)\big\|_{X_0} + (\delta+\delta')\big(\|v\|_{C_\alpha^\alpha((0,1],X_0)} + \|v\|_{C^\alpha([1,T_*),X_0,-\omega)}\big)\big].$$

Thus, if δ and δ' are sufficiently small then we have

$$\|v\|_{C_\alpha^\alpha((0,1],X_0)} + \|v\|_{C^\alpha([1,T_*),X_0,-\omega)} \leqslant C\big\|v(0)\big\|_{X_0},$$

which implies, in particular, that

$$\big\|v(t)\big\|_{X_0} \leqslant Ce^{-\omega t}\big\|v(0)\big\|_{X_0} \quad \text{for } 0 \leqslant t < T_*, \tag{2.10}$$

where C is independent of T_*. Next, since $\mathcal{N}_1(x,\varphi(x)) = 0$, we have

$$x'(t) = \mathcal{N}_1\big(x(t), y(t)\big) - \mathcal{N}_1\big(x(t), \varphi\big(x(t)\big)\big) \equiv \mathcal{G}_1(t).$$

It can be easily shown that

$$\big\|\mathcal{G}_1(t)\big\|_X \leqslant C\big(\big\|u(t)\big\|_{X_0} + \big\|\tilde{u}(t)\big\|_{X_0}\big)\big\|v(t)\big\|_{X_0}.$$

Hence

$$\big\|x(t)\big\|_{X_0} \leqslant C\big\|x(t)\big\|_X \leqslant C\int_0^t \big\|\mathcal{G}_1(s)\big\|_X\,ds \leqslant C(\delta+\delta')\int_0^t \big\|v(s)\big\|_{X_0}\,ds \leqslant C\big\|v(0)\big\|_{X_0}. \tag{2.11}$$

In getting the first inequality we used the fact that when restricted on $PX = PX_0 = \operatorname{Ker} A$, the two norms $\|\cdot\|_{X_0}$ and $\|\cdot\|_X$ are equivalent because the dimension of this space is finite. Now, since $u(t) = x(t) + v(t) + \varphi(x(t))$ and $y(t) = v(t) + \varphi(x(t))$, by using (2.10) and (2.11) we can easily deduce that if $\mathcal{O}_2$ is sufficiently small then for any $u_0 \in \mathcal{O}_2$ we have $T_* = T^*(u_0) = \infty$. This proves the assertion (1).

Similarly as in the proof of (2.11), for any $s > t \geqslant 0$ we have

$$\big\|x(t) - x(s)\big\|_{X_0} \leqslant C\int_t^s \big\|\mathcal{G}_1(\tau)\big\|_X\,d\tau \leqslant C\int_t^s \big\|v(\tau)\big\|_{X_0}\,d\tau \leqslant C\big(e^{-\omega t} - e^{-\omega s}\big)\big\|v(0)\big\|_{X_0}. \tag{2.12}$$

Hence $\lim_{t\to\infty} x(t)$ exists. Let $\bar{x} = \lim_{t\to\infty} x(t)$ and $\bar{u} = \bar{x} + \varphi(\bar{x})$. Then $\bar{u} \in \mathcal{M}^c$, so that it is a stationary point of the equation $u' = F(u)$. Moreover, by the facts that $\lim_{t\to\infty} x(t) = \bar{x}$ and $\lim_{t\to\infty} v(t) = 0$ in $(I-P)X_0$ we see that $\lim_{t\to\infty} u(t) = \bar{u}$ in X_0. Letting $s \to \infty$ in (2.12) we see that

$$\big\|x(t) - \bar{x}\big\|_{X_0} \leqslant Ce^{-\omega t}\big\|v(0)\big\|_{X_0}. \tag{2.13}$$

From (2.10) and (2.13) we obtain

$$\big\|u(t) - \bar{u}\big\|_{X_0} \leqslant Ce^{-\omega t}\big\|v(0)\big\|_{X_0}. \tag{2.14}$$

Hence $\mathcal{M}^c$ is the unique center manifold of the equation $u' = F(u)$ in a neighborhood of the origin. This proves the assertion (2).

Next we consider the assertion (iii). We follow the idea in the proof of Theorem 9.1.4(ii) of [29] to prove this assertion. To this end we rewrite the equation $u' = F(u)$ into the form $u' = Au + N(u)$.

Clearly, $N \in C^{2-0}(\mathcal{O}_1, X)$, $N(0)=0$, and $N'(0)=0$, so that $\|N(u)\|_X \leqslant C\|u\|_{X_0}^2$ and $\|N(u)-N(v)\|_X \leqslant C(\|u\|_{X_0}+\|v\|_{X_0})\|u-v\|_{X_0}$. Given $y \in B_2(0,\delta')$, we consider the following problem:

$$\begin{cases} u'(t) = Au(t) + N\big(u(t)\big) \quad \text{for } t>0, \\ (I-P)u(0) = y \quad \text{and} \quad \lim\limits_{t\to\infty} \big\|u(t)\big\|_{X_0} = 0. \end{cases} \tag{2.15}$$

We assert that this problem has a unique solution provided δ' is sufficiently small. To prove existence let α and ω be as before, and for a positive number R to be specified later we introduce a metric space $(M_\omega^\alpha(R), d)$ by letting

$$M_\omega^\alpha(R) = \big\{u \in C\big((0,\infty), X_0\big) \cap C_\alpha^\alpha\big((0,1], X_0\big) \cap C^\alpha\big([1,\infty), X_0, -\omega\big)\colon\ |||u||| \leqslant R\big\},$$

where

$$|||u||| = \|u\|_{C_\alpha^\alpha((0,1],X_0)} + \|u\|_{C^\alpha([1,\infty),X_0,-\omega)},$$

and $d(u,v) = |||u-v|||$. We define a mapping $\Psi_y : M_\omega^\alpha(R) \to C((0,\infty), X_0)$ by letting $\Psi_y(u) = v$ for every $u \in M_\omega^\alpha(R)$, where

$$v(t) = e^{tA_-}y + \int_0^t e^{(t-s)A_-}(I-P)N\big(u(s)\big)\,ds - \int_t^\infty PN\big(u(s)\big)\,ds.$$

Using Lemma 2.2, we can easily prove that for sufficiently small R, δ' and for any $y \in B_2(0,\delta')$, Ψ_y is well defined, maps $M_\omega^\alpha(R)$ into itself and is a contraction mapping. Hence, Ψ_y has a unique fixed point in $M_\omega^\alpha(R)$ which we denote by u_y. Since $AP=0$ so that $e^{(t-s)A}P = P$, it can be easily verified that u_y is a solution of (2.15). This proves existence. To prove uniqueness, for any $(x,y) \in B_1(0,\delta) \times B_2(0,\delta')$ we denote by $u(t,x,y)$ the unique solution of the equation $u' = F(u)$ satisfying the initial conditions $Pu(0) = x$ and $(I-P)u(0) = y$. By assertion (1) we know that $u(t,x,y)$ exists for all $t \geqslant 0$. Using the fact that $AP=0$ we can easily deduce that $\lim_{t\to\infty} u(t,x,y) = 0$ if and only if

$$x + \int_0^\infty PN\big(u(s,x,y)\big)\,ds = 0. \tag{2.16}$$

We introduce a mapping $\mathcal{F} : B(0,\delta) \times B(0,\delta') \to PX$ by letting

$$\mathcal{F}(x,y) = x + \int_0^\infty PN\big(u(s,x,y)\big)\,ds$$

for $(x,y) \in B_1(0,\delta) \times B_2(0,\delta')$. $\mathcal{F}$ is well defined. Indeed, we know that for any $(x,y) \in B_1(0,\delta) \times B_2(0,\delta')$, $\bar{u} = \lim_{t\to\infty} u(t,x,y)$ exists and it belongs to $\mathcal{M}^c$. Let $\bar{x} = P\bar{u}$ and $\bar{y} = (I-P)\bar{u}$. Then $\bar{y} = \varphi(\bar{x})$, so that $PN(\bar{u}) = \mathcal{F}_1(\bar{x},\bar{y}) = \mathcal{F}_1(\bar{x},\varphi(\bar{x})) = 0$. Thus we have

$$\big\|PN\big(u(t,x,y)\big)\big\|_X = \big\|PN\big(u(t,x,y)\big) - PN(\bar{u})\big\|_X \leqslant C\big\|u(t,x,y) - \bar{u}\big\|_{X_0} \leqslant C(x,y)e^{-\omega t}.$$

Hence, the integral in the definition of $\mathcal{F}$ is convergent. We assert that $\mathcal{F} \in C^{2-0}(B(0,\delta) \times B(0,\delta'), PX)$. Indeed, since $F \in C^{2-0}(\mathcal{O}, X)$, by a standard result we have that the mapping $u_0 \to$

$u(\cdot, u_0)$ belongs to $C^{2-0}(\mathcal{O}_1, C([0,T], X) \cap C_\alpha^\alpha((0,T], X_0))$ for any given $T > 0$, cf. Section 8.3 of [29].[1] Moreover, letting $V(t) = D_{u_0}u(t, u_0)$ $(\in C([0,\infty), L(X_0, X)) \cap C((0,\infty), L(X_0)) \cap C^1((0,\infty), L(X_0, X)))$ and $A(t) = F'(u(t, u_0))$ $(\in C((0,\infty), L(X_0, X)))$, we have that $V(t)$ is the solution of the problem

$$V'(t) = A(t)V(t) \quad \text{for } t > 0, \quad \text{and} \quad V(0) = \mathrm{id}.$$

Since $F'(0)$ is a sectorial operator in X (with domain X_0), $\operatorname{Re}\lambda \leqslant 0$ for any $\lambda \in \sigma(F'(0))$, and $u(t, u_0)$ lies in $\mathcal{O}_1$ for all $t \geqslant 0$, by taking $\mathcal{O}_1$ sufficiently small we see that for any $t \geqslant 0$, $A(t) = F'(u(t, u_0))$ is a sectorial operator in X (with domain X_0), and $\operatorname{Re}\lambda \leqslant (1/2)\omega$ for any $\lambda \in \sigma(A(t))$. Using these facts and the standard theory for linear parabolic differential equations in Banach spaces (cf. Section 5.8 of [31]) we can deduce that $\|V(t)\|_{L(X_0)} \leqslant Ce^{\frac{1}{2}\omega t}$ for all $t \geqslant 0$. Besides, by a similar argument as above we can prove that $\|PN'(u(t,x,y))\|_{L(X_0,X)} \leqslant C(x,y)e^{-\omega t}$ for $t \geqslant 0$. It follows that the integrals

$$\int_0^\infty PN'\big(u(s,x,y)\big)D_x u(s,x,y)\,ds \quad \text{and} \quad \int_0^\infty PN'\big(u(s,x,y)\big)D_y u(s,x,y)\,ds$$

are both convergent. Hence, $\mathcal{F}(x,y)$ is Fréchet differentiable in (x,y) and we have

$$D_x\mathcal{F}(x,y) = \mathrm{id} + \int_0^\infty PN'\big(u(s,x,y)\big)D_x u(s,x,y)\,ds,$$

$$D_y\mathcal{F}(x,y) = \int_0^\infty PN'\big(u(s,x,y)\big)D_y u(s,x,y)\,ds.$$

Using these expressions we can further prove that $D_x\mathcal{F}(x,y)$ and $D_y\mathcal{F}(x,y)$ are Lipschitz continuous. This proves the desired assertion. Now, since $u(t,0,0) = 0$ and $N(0) = N'(0) = 0$, we have $\mathcal{F}(0,0) = 0$ and $D_x\mathcal{F}(0,0) = \mathrm{id}$. Thus, by the implicit function theorem we conclude that the solution of (2.16) is unique for fixed $y \in B_2(0,\delta')$, provided δ' is sufficiently small. This proves uniqueness.

We now introduce a mapping $\psi : (I-P)X_0 \to PX$ by

$$\psi(y) = Pu_y(0) = -\int_0^\infty PN\big(u_y(s)\big)\,ds \quad \text{for } y \in B_2(0,\delta').$$

Clearly, $x = \psi(y)$ is the implicit function solving the equation $\mathcal{F}(x,y) = 0$, so that $\psi \in C^{2-0}(B_2(0,\delta'), PX)$. Letting $\mathcal{M}^s = \operatorname{graph}\psi$, we see that all requirements of the assertion (3) are satisfied. This proves the assertion (3).

Finally, given $u_0 \in \mathcal{O}_3$ let $\bar{u}$ be as in (2.14). Since $\bar{u} \in \mathcal{M}^c$, there exists a unique $\sigma \in G$ such that $S_\sigma(0) = \bar{u}$. Let $v_0 = S_{\sigma^{-1}}(u_0) = S_\sigma^{-1}(u_0)$. Then we have

$$\lim_{t\to\infty} u(t, v_0) = \lim_{t\to\infty} S_\sigma^{-1}\big(u(t,u_0)\big) = S_\sigma^{-1}\big(S_\sigma(0)\big) = 0,$$

so that $v_0 \in \mathcal{M}^s$. Noticing that $u_0 = S_\sigma(v_0)$ and (2.4) is an immediate consequence of (2.14), we get the assertion (4). This completes the proof. $\square$

[1] In Section 8.3 of [29] the compatibility condition $F(u_0) \in \overline{X}_0$ is assumed. By a similar reason as in the introduction of Section 9.1 we see that this condition can be removed when the space $C([0,T], X_0) \cap C_\alpha^\alpha((0,T], X_0)$ there is replaced with the space $C([0,T], X) \cap C_\alpha^\alpha((0,T], X_0)$ which we use here.

Remark 2.1. Checking the proof of Theorem 2.1, we see that the condition on the Lie group action p can be weakened, that is, p need not to act on the space X; an action on X_0 is sufficient.

3. Reduction of the problem

In this section we shall reduce the problem (1.1)–(1.7) into an initial value problem of an abstract differential equation in some Banach space. The reduction will be fulfilled in two steps: First we use the Hanzawa transformation to convert the free boundary problem (1.1)–(1.7) into an initial–boundary value problem on the fixed domain Ω_s. Next we solve the equations for p in terms of σ and ρ, the function defining the free boundary $\partial\Omega(t)$, to reduce this initial–boundary value problem into a purely evolutionary type and regard it as a differential equation in a suitable Banach space, which will be the desired abstract equation. For simplicity of notation, later on we always assume that $R_s = 1$. Note that this assumption is reasonable because the general case can be reduced into this special case by making suitable rescaling. It follows that

$$\Omega_s = \mathbb{B}^n = \{x \in \mathbb{R}^n\colon\ |x| < 1\} \quad \text{and} \quad \partial\Omega_s = \partial\mathbb{B}^n = \mathbb{S}^{n-1}.$$

Besides, throughout this paper we assume that the initial domain Ω_0 is a small perturbation of $\Omega_s = \mathbb{B}^n$, so that $\partial\Omega_0$ is contained in a small neighborhood of $\partial\Omega_s = \mathbb{S}^{n-1}$.

To perform the first step of reduction let us first consider the Hanzawa transformation.

Fix a positive number δ such that $0 < \delta < 1$, and denote

$$\mathcal{O}_\delta(\mathbb{S}^{n-1}) = \{\rho \in C^1(\mathbb{S}^{n-1})\colon\ \|\rho\|_{C^1(\mathbb{S}^{n-1})} < \delta\}.$$

Given $\rho \in \mathcal{O}_\delta(\mathbb{S}^{n-1})$, we define a mapping $\theta_\rho : \mathbb{S}^{n-1} \to \mathbb{R}^n$ by letting $\theta_\rho(\xi) = (1 + \rho(\xi))\xi$ for $\xi \in \mathbb{S}^{n-1}$, and denote

$$\Gamma_\rho = \operatorname{Im}(\theta_\rho) = \{x \in \mathbb{R}^n\colon\ x = (1 + \rho(\xi))\xi,\ \xi \in \mathbb{S}^{n-1}\}.$$

Clearly, Γ_ρ is a closed C^1-hypersurface diffeomorphic to $\mathbb{S}^{n-1}$, and θ_ρ is a C^1-diffeomorphism from $\mathbb{S}^{n-1}$ onto Γ_ρ. We denote by Ω_ρ the domain enclosed by Γ_ρ. In the following we always assume that $\partial\Omega_0$ is of C^1 class and is contained in the δ-neighborhood of $\mathbb{S}^{n-1}$. More precisely, we assume that there exists $\rho_0 \in \mathcal{O}_\delta(\mathbb{S}^{n-1})$ such that $\partial\Omega_0 = \Gamma_{\rho_0}$, and, accordingly, $\Omega_0 = \Omega_{\rho_0}$.

Let m be an integer, $m \geqslant 2$, and let $n/(m-1) < q < \infty$. Then we have $B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \subseteq C^1(\mathbb{S}^{n-1})$. The well-known trace theorem ensures that the trace operator $\operatorname{tr}(u) = u|_{\mathbb{S}^{n-1}}$ from $C^\infty(\bar{\mathbb{B}}^n)$ to $C^\infty(\mathbb{S}^{n-1})$ can be extended to $W^{m,q}(\mathbb{B}^n)$ such that it maps $W^{m,q}(\mathbb{B}^n)$ into $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ and is bounded and surjective. We introduce a right inverse Π of this operator as follows: Given $\rho \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$, let $u \in W^{m,q}(\mathbb{B}^n)$ be the unique solution of the boundary value problem

$$\Delta u = 0 \quad \text{in } \mathbb{B}^n, \quad \text{and} \quad u = \rho \quad \text{on } \mathbb{S}^{n-1},$$

and define $\Pi(\rho) = u$. Then clearly $\operatorname{tr}(\Pi(\rho)) = \rho$ for $\rho \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$, and the standard L^p estimate and the maximum principle yield the following relations:

$$\|\Pi(\rho)\|_{W^{m,q}(\mathbb{B}^n)} \leqslant C\|\rho\|_{B_{qq}^{m-1/q}(\mathbb{S}^{n-1})} \quad \text{and} \quad \sup_{x\in\mathbb{B}^n}|\Pi(\rho)(x)| = \max_{x\in\mathbb{S}^{n-1}}|\rho(x)|.$$

Note that since $W^{m,q}(\mathbb{B}^n) \hookrightarrow C^1(\bar{\mathbb{B}}^n)$, the first relation implies that

$$\|\Pi(\rho)\|_{C^1(\bar{\mathbb{B}}^n)} \leqslant C_0\|\rho\|_{B_{qq}^{m-1/q}(\mathbb{S}^{n-1})}. \tag{3.1}$$

Here we use the special notation C_0 to denote the constant in (3.1) because later on this constant will play a special role. We now introduce

$$\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}) = \{\rho \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1}):\ \|\rho\|_{B_{qq}^{m-1/q}(\mathbb{S}^{n-1})} < \delta,\ \|\rho\|_{C^1(\mathbb{S}^{n-1})} < \delta\}.$$

In the sequel we further assume that $\delta < \min\{1/5, (3C_0)^{-1}\}$. Take a function $\phi \in C^\infty(\mathbb{R}, [0,1])$ such that

$$\phi(\tau) = 1 \quad \text{for } |\tau| \leqslant \delta, \quad \phi(\tau) = 0 \quad \text{for } |\tau| \geqslant 3\delta, \quad \text{and} \quad \sup|\phi'| < \frac{2}{3}\delta^{-1}.$$

Given $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$, we define the *Hanzawa transformation* $\Theta_\rho : \overline{\mathbb{B}}^n \to \overline{\Omega}_\rho$ by

$$\Theta_\rho(x) = x + \phi(|x| - 1)\Pi(\rho)(x)\omega(x) \quad \text{for } x \in \overline{\mathbb{B}}^n,$$

where $\omega(x) = x/|x|$ for $x \in \mathbb{R}^n \setminus \{0\}$, and $\omega(0) = 0$. The choice of δ and the inequality (3.1) ensures that for fixed $\omega \in \mathbb{S}^{n-1}$, the function $r \to r + \phi(r-1)\Pi(\rho)(r\omega)$ is strictly monotone increasing for $0 \leqslant r \leqslant 1$, so that Θ_ρ is a bijection from $\overline{\mathbb{B}}^n$ onto $\overline{\Omega}_\rho$. In fact, since the derivative of this function is strictly positive, it can be easily shown that $\Theta_\rho \in W^{m,q}(\mathbb{B}^n, \Omega_\rho)$ and $\Theta_\rho^{-1} \in W^{m,q}(\Omega_\rho, \mathbb{B}^n)$. Besides, it is clear that $\Theta_\rho|_{\mathbb{S}^{n-1}} = \theta_\rho$. Since $W^{m,q}(\mathbb{B}^n) \subseteq C^1(\overline{\mathbb{B}}^n)$ and $W^{m,q}(\Omega_\rho) \subseteq C^1(\overline{\Omega}_\rho)$, we see that Θ_ρ is particularly a C^1-diffeomorphism from $\overline{\mathbb{B}}^n$ onto $\overline{\Omega}_\rho$.

As usual we denote by Θ_*^ρ and Θ_ρ^* respectively the push-forward and pull-back operators induced by Θ_ρ, i.e., $\Theta_*^\rho u = u \circ \Theta_\rho^{-1}$ for $u \in C(\overline{\mathbb{B}}^n)$, and $\Theta_\rho^* u = u \circ \Theta_\rho$ for $u \in C(\overline{\Omega}_\rho)$. Similarly, θ_ρ^* denotes the pull-back operator induced by θ_ρ, i.e., $\theta_\rho^* u(\xi) = u(\theta_\rho(\xi))$ for $u \in C(\Gamma_\rho)$ and $\xi \in \mathbb{S}^{n-1}$. Later, we shall need the following result:

Lemma 3.1. *Let m be an integer and $1 \leqslant q < \infty$. Let Ω_1 and Ω_2 be two open subsets of $\mathbb{R}^n$. Let Φ be a diffeomorphism from Ω_1 to Ω_2 such that $\Phi \in W^{m,q}(\Omega_1, \mathbb{R}^n)$ and $\Phi^{-1} \in W^{m,q}(\Omega_2, \mathbb{R}^n)$. Assume that $m \geqslant 2$ and $q > n/(m-1)$. Then for any $0 \leqslant k \leqslant m$ we have*

$$\Phi_* \in L(W^{k,q}(\Omega_1), W^{k,q}(\Omega_2)) \quad \textit{and} \quad \Phi^* \in L(W^{k,q}(\Omega_2), W^{k,q}(\Omega_1)).$$

In particular, for any $\rho \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ and $0 \leqslant k \leqslant m$ we have

$$\Theta_*^\rho \in L(W^{k,q}(\mathbb{B}^n), W^{k,q}(\Omega_\rho)) \quad \textit{and} \quad \Theta_\rho^* \in L(W^{k,q}(\Omega_\rho), W^{k,q}(\mathbb{B}^n)).$$

Proof. See the proof of Lemma 2.1 of [14] for the case $k = m$. Proofs for the rest cases $0 \leqslant k \leqslant m-1$ are similar and simpler. $\square$

Next we introduce some notations.

In the sequel we assume that $m \geqslant 2$ and $q > n/(m-1)$. As in [14], for $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$ we introduce a second-order partial differential operator $\mathcal{A}(\rho) : W^{m,q}(\mathbb{B}^n) \to W^{m-2,q}(\mathbb{B}^n)$ by

$$\mathcal{A}(\rho)u = \Theta_\rho^* \Delta(\Theta_*^\rho u) \quad \text{for } u \in W^{m,q}(\mathbb{B}^n).$$

By Lemma 3.1 we see that $\mathcal{A}(\rho) \in L(W^{m,q}(\mathbb{B}^n), W^{m-2,q}(\mathbb{B}^n))$. We also introduce nonlinear operators $\mathcal{F}$ and $\mathcal{G} : W^{m,q}(\mathbb{B}^n) \to W^{m,q}(\mathbb{B}^n)$ respectively by

$$\mathcal{F}(u) = f \circ u, \qquad \mathcal{G}(u) = g \circ u \quad \text{for } u \in W^{m,q}(\mathbb{B}^n).$$

Since the condition $q > n/(m-1) > n/m$ implies that $W^{m,q}(\mathbb{B}^n)$ is an algebra, we see that these definitions make sense and we have $\mathcal{F}, \mathcal{G} \in C^\infty(W^{m,q}(\mathbb{B}^n), W^{m,q}(\mathbb{B}^n))$. Given $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$ we denote

$$\psi_\rho(x) = |x| - 1 - \rho\big(\omega(x)\big) \quad \text{for } x \in \mathcal{R} \equiv \big\{x \in \mathbb{R}^n\colon\ 1 - 4\delta < |x| < 1 + 4\delta\big\}.$$

Clearly, $\psi_\rho \in B_{qq}^{m-1/q}(\mathcal{R})$. Since $\Gamma_\rho = \{x \in \mathcal{R}\colon\ \psi_\rho(x) = 0\}$, we see that the unit outward normal field $\mathbf{n}$ on Γ_ρ is given by $\mathbf{n}(x) = \nabla\psi_\rho(x)/|\nabla\psi_\rho(x)|$ for $x \in \Gamma_\rho$. We introduce a first-order trace operator $\mathcal{D}(\rho) : W^{m,q}(\mathbb{B}^n) \to B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1})$ by

$$\mathcal{D}(\rho)u = \theta_\rho^*\big(\operatorname{tr}_{\Gamma_\rho}\big(\nabla\big(\Theta_*^\rho u\big) \cdot \nabla\psi_\rho\big)\big) \quad \text{for } u \in W^{m,q}\big(\mathbb{B}^n\big),$$

where $\operatorname{tr}_{\Gamma_\rho}$ denotes the usual trace operator from $\overline{\Omega}_\rho \cap \mathcal{R}$ to Γ_ρ, i.e., $\operatorname{tr}_{\Gamma_\rho}(u) = u|_{\Gamma_\rho}$ for $u \in C(\overline{\Omega}_\rho \cap \mathcal{R})$. It can be easily seen that $\mathcal{D}(\rho)$ maps $W^{m,q}(\mathbb{B}^n)$ into $B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1})$, and $\mathcal{D}(\rho) \in L(W^{m,q}(\mathbb{B}^n), B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1}))$ for any $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$. Similarly, given $(\rho, u) \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}) \times W^{m,q}(\mathbb{B}^n)$, we introduce a first-order pseudo-differential operator $\mathcal{P}(\rho, u)\colon W^{m,q}(\mathbb{B}^n) \to W^{m-1,q}(\mathbb{B}^n)$ as follows:

$$\mathcal{P}(\rho, u)v = \mathcal{M}(\rho, u) \cdot \Pi\big(\mathcal{D}(\rho)v\big) \quad \text{for } v \in W^{m,q}\big(\mathbb{B}^n\big).$$

Here we used the same notation Π as before to denote the bounded right inverse of the trace operator $\operatorname{tr} : W^{m-1,q}(\mathbb{B}^n) \to B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1})$ such that its restriction on $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ is equal to the previous Π, and

$$\mathcal{M}(\rho, u)(x) = \phi\big(|x| - 1\big)\big\langle\big(\Theta_\rho^*\nabla\Theta_*^\rho u\big)(x), \omega(x)\big\rangle \quad \text{for } x \in \mathbb{B}^n,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. We note that $\mathcal{M}(\rho, u) \in W^{m-1,q}(\mathbb{B}^n)$ and the mapping $u \to \mathcal{M}(\rho, u)$ is a first-order partial differential operator. Since $[v \to \Pi(\mathcal{D}(\rho)v)] \in L(W^{m,q}(\mathbb{B}^n), W^{m-1,q}(\mathbb{B}^n))$ and the condition $q > n/(m-1)$ implies that $W^{m-1,q}(\mathbb{B}^n)$ is an algebra, we see that $\mathcal{P}(\rho, u) \in L(W^{m,q}(\mathbb{B}^n), W^{m-1,q}(\mathbb{B}^n))$. Finally, we define the transformed mean curvature operator $\mathcal{K} : C^2(\mathbb{S}^{n-1}) \cap \mathcal{O}_\delta(\mathbb{S}^{n-1}) \to C(\mathbb{S}^{n-1})$ by

$$\mathcal{K}(\rho) = \theta_\rho^*(\kappa_{\Gamma_\rho}),$$

where κ_{Γ_ρ} denotes the mean curvature of the hypersurface Γ_ρ (recall that $\kappa_{\Gamma_\rho} \in C(\Gamma_\rho, \mathbb{R})$ for C^2 class hypersurface Γ_ρ). Later we shall restrict $\mathcal{K}$ in $\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$ and shall see that $\mathcal{K} \in C^\infty(\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}), B_{qq}^{m-2-1/q}(\mathbb{S}^{n-1}))$.

Let T be a given positive number and consider a function $\rho : [0, T] \to \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$. We assume that $\rho \in C([0, T], \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}))$. Given such a ρ, we denote

$$\Gamma_\rho(t) = \Gamma_{\rho(t)}, \qquad \Omega_\rho(t) = \Omega_{\rho(t)} \quad (0 \leqslant t \leqslant T).$$

Later on in case no confusion can be produced we shall occasionally abbreviate $\Gamma_\rho(t)$ and $\Omega_\rho(t)$ respectively as Γ_ρ and Ω_ρ. We shall briefly write the families of operators $t \to \mathcal{A}(\rho(t))$ and $t \to \mathcal{D}(\rho(t))$ $(0 \leqslant t \leqslant T)$ as $\mathcal{A}(\rho)$ and $\mathcal{D}(\rho)$, respectively, and for $u, v : [0, T] \to W^{m,q}(\mathbb{B}^n)$, we briefly write the families of functions $\mathcal{F}(\rho(t), u(t))$, $\mathcal{G}(\rho(t), u(t))$ and $\mathcal{M}(\rho(t), u(t))v(t)$ $(0 \leqslant t \leqslant T)$ respectively as $\mathcal{F}(\rho, u)$, $\mathcal{G}(\rho, u)$ and $\mathcal{M}(\rho, u)v$. Besides, we shall identify a function $\rho : [0, T] \to C(\mathbb{S}^{n-1})$ (resp. $u : [0, T] \to C(\overline{\mathbb{B}}^n)$) with the corresponding function on $\mathbb{S}^{n-1} \times [0, T]$ (resp. $\overline{\mathbb{B}}^n \times [0, T]$) defined by $\rho(\xi, t) = \rho(t)(\xi)$ (resp. $u(x, t) = u(t)(x)$), where $t \in [0, T]$ and $\xi \in \mathbb{S}^{n-1}$ (resp. $x \in \overline{\mathbb{B}}^n$), and vice versa.

With the above notations, it is not hard to verify that if we denote

$$u(x, t) = \sigma\big(\Theta_{\rho(t)}(x), t\big), \qquad v(x, t) = p\big(\Theta_{\rho(t)}(x), t\big),$$

then the Hanzawa transformation transforms (1.1)–(1.7) into the following system of equations:

$$c\partial_t u - \mathcal{A}(\rho)u + c\mathcal{P}(\rho, u)v = -\mathcal{F}(u) \quad \text{in } \mathbb{B}^n \times (0, T], \tag{3.2}$$

$$-\mathcal{A}(\rho)v = \mathcal{G}(u) \quad \text{in } \mathbb{B}^n \times (0, T], \tag{3.3}$$

$$u = \bar{\sigma} \quad \text{on } \mathbb{S}^{n-1} \times (0, T], \tag{3.4}$$

$$v = \gamma\mathcal{K}(\rho) \quad \text{on } \mathbb{S}^{n-1} \times (0, T], \tag{3.5}$$

$$\partial_t \rho + \mathcal{D}(\rho)v = 0 \quad \text{on } \mathbb{S}^{n-1} \times (0, T], \tag{3.6}$$

$$u(0) = u_0 \quad \text{on } \mathbb{B}^n, \tag{3.7}$$

$$\rho(0) = \rho_0 \quad \text{on } \mathbb{S}^{n-1}, \tag{3.8}$$

where $u_0 = \Theta^*_{\rho_0}\sigma_0$. Indeed, it is immediate to see that (3.3), (3.4), (3.5), (3.7) and (3.8) are transformations of (1.2), (1.3), (1.4), (1.6) and (1.7), respectively. For the proof that the transformation of (1.5) is (3.6), we refer the reader to see the deduction of (2.19) in [14] and (2.8) in [20]. Finally, (3.2) is obtained from transforming (1.1) and using (3.6).

To establish properties of the operator $\mathcal{K}$, we need the following lemma:

Lemma 3.2. (i) *Let k, m be nonnegative integers, and $p, q \in [1, \infty]$. Let Ω be an open subset of $\mathbb{R}^n$ with a smooth boundary. Assume that $k \geqslant m$ and either $p \leqslant n/m$, $k > n/q$ or $p > n/m$, $k - n/q \geqslant m - n/p$. Then we have*

$$\|uv\|_{W^{m,p}(\Omega)} \leqslant C\|u\|_{W^{k,q}(\Omega)}\|v\|_{W^{m,p}(\Omega)}. \tag{3.9}$$

(ii) *Let $s, t > 0$ and $p, q, r_1, r_2 \in [1, \infty]$. Let Ω be as before. Assume that $t \geqslant s$ and either $p \leqslant n/s$, $t > n/q$ or $p > n/s$, $t - n/q \geqslant s - n/p$. Then we have*

$$\|uv\|_{B^s_{pr_1}(\Omega)} \leqslant C\|u\|_{B^t_{qr_2}(\Omega)}\|v\|_{B^s_{pr_1}(\Omega)}. \tag{3.10}$$

Here r_1, r_2 are arbitrary numbers in $[1, \infty]$ in case $t > s$, and $1 \leqslant r_2 \leqslant r_1 \leqslant \infty$ if $t = s$.

Proof. To prove (3.9), we note that since $k > n/q$, there holds $W^{k,q}(\Omega) \hookrightarrow L^\infty(\Omega)$. Hence

$$\|uv\|_{L^p(\Omega)} \leqslant \|u\|_{L^\infty(\Omega)}\|v\|_{L^p(\Omega)} \leqslant C\|u\|_{W^{k,q}(\Omega)}\|v\|_{W^{m,p}(\Omega)}. \tag{3.11}$$

Next let $\alpha \in \mathbb{Z}^n_+$ be an arbitrary n-index of length m, i.e., $|\alpha| = m$. We write the Leibnitz formula:

$$\partial^\alpha(uv) = \sum_{\beta \leqslant \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!}\partial^\beta u \partial^{\alpha-\beta}v.$$

For every n-index $\beta \leqslant \alpha$ we take $r_1, r_2 \in [1, \infty]$ as follows:

$$\begin{cases} \dfrac{1}{r_1} = \dfrac{1}{q} - \dfrac{k - |\beta|}{n}, \quad \dfrac{1}{r_2} = \dfrac{1}{p} - \dfrac{1}{q} + \dfrac{k - |\beta|}{n} \quad \text{if } |\beta| > k - \dfrac{n}{q}, \\ r_1 = \dfrac{1}{\varepsilon}, \quad \dfrac{1}{r_2} = \dfrac{1}{p} - \varepsilon \quad \text{if } |\beta| = k - \dfrac{n}{q}, \\ r_1 = \infty, \quad r_2 = p \quad \text{if } |\beta| < k - \dfrac{n}{q}, \end{cases}$$

where ε is a small positive number. Note that since $|\beta| \leqslant m \leqslant k$, we have $\frac{1}{p} - \frac{1}{q} + \frac{k-|\beta|}{n} \geqslant 0$. Clearly $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{p}$, $\|\partial^\beta u\|_{L^{r_1}(\Omega)} \leqslant C\|u\|_{W^{k,q}(\Omega)}$ and $\|\partial^{\alpha-\beta} v\|_{L^{r_2}(\Omega)} \leqslant C\|u\|_{W^{m,p}(\Omega)}$. Hence

$$\big\|\partial^\alpha(uv)\big\|_{L^p(\Omega)} \leqslant C\sum_{\beta\leqslant\alpha}\big\|\partial^\beta u\big\|_{L^{r_1}(\Omega)}\big\|\partial^{\alpha-\beta} v\big\|_{L^{r_2}(\Omega)} \leqslant C\|u\|_{W^{k,q}(\Omega)}\|v\|_{W^{m,p}(\Omega)}. \tag{3.12}$$

Combining (3.11) and (3.12), we get (3.9).

To prove (3.10), we need the following extended version of (3.9):

$$\|uv\|_{H^s_p(\Omega)} \leqslant C\|u\|_{H^t_q(\Omega)}\|v\|_{H^s_p(\Omega)}, \tag{3.13}$$

where $H^s_p(\Omega)$ and $H^t_q(\Omega)$ are fractional or generalized Sobolev spaces of indices (s,p) and (t,q), respectively (cf. Section 6.2 of [5] for the definition), and s, t, p and q are real numbers satisfying the conditions in the assertion (ii). In the case $\Omega = \mathbb{R}^n$, (3.13) can be proved by using the following well-known inequality for fractional derivatives:

$$\big\|J^s(uv)\big\|_p \leqslant C\big(\big\|J^s u\big\|_{r_1}\|v\|_{r_2} + \|u\|_{r_3}\big\|J^s v\big\|_{r_4}\big),$$

where J^s denotes the Riesz potential operator, i.e., $J^s u = F^{-1}((1+|\xi|^2)^{\frac{s}{2}}\tilde{u}(\xi))$ (for $u \in S'(\mathbb{R}^n)$), and the indices p and r_j's satisfy the relations $1/p = 1/r_1 + 1/r_2 = 1/r_3 + 1/r_4$ (cf. [27, Appendix]). Given s, t, p and q satisfying the conditions in the assertion (ii), we use a similar idea as in the proof of (3.12) to choose indices r_j's. Then (3.13) follows. In the case that Ω is a general smooth domain in $\mathbb{R}^n$, (3.13) follows by using the extension operator from Ω to $\mathbb{R}^n$. This proves (3.13). Now, given $p, q \in [1,\infty]$, we denote by Δ the set of all pairs (s,t) of positive numbers satisfying the assumptions in the assertion (ii). Clearly, Δ is the upper unbounded wedge set in the (s,t)-plane enclosed by the lower boundary line $t = n/q$, the left boundary line $s = 0$ and the right boundary line $t - n/q = s - n/p$ (in case $n/q > n/p$) or $t = s$ (in case $n/q \leqslant n/p$); it is open on the left and lower boundaries and closed on the right boundary. The fact that Δ is open on the left and lower boundaries and the slope of the right boundary line is positive ensures that for any given $(s,t) \in \Delta$ we can find $(s_1,t_1), (s_2,t_2) \in \Delta$ such that $s_1 < s < s_2$, $t_1 < t < t_2$ and these three points lie in the same line. Replacing (s,t) in (3.13) with (s_1,t_1) and (s_2,t_2) respectively, we get two inequalities expressed in forms of fractional Sobolev spaces, and interpolating these two inequalities we obtain (3.10). □

Corollary 3.3. *Assume that $m \geqslant 2$, $q \geqslant 1$ and either $0 < s \leqslant m-1-1/q$ and $q > n/(m-1)$ or $-1/q \leqslant s \leqslant 0$ and either $n \geqslant m$, $q > n/(m-1)$ or $n < m$, $2n/(m+n-2) \leqslant q < n/(n-1)$. Then we have*

$$\|uv\|_{B^s_{qq}(\mathbb{S}^{n-1})} \leqslant C\|u\|_{B^{m-1-1/q}_{qq}(\mathbb{S}^{n-1})}\|v\|_{B^s_{qq}(\mathbb{S}^{n-1})}. \tag{3.14}$$

Proof. If $s > 0$ then the desired assertion follows immediately from Lemma 3.2(ii), because we can easily verify that all conditions of Lemma 3.2(ii) are satisfied when we replace t with $m-1-1/q$, p with q and n with $n-1$. Next we consider the case $-1/q \leqslant s \leqslant 0$. We can also easily verify that in this case all conditions of Lemma 3.2(ii) are satisfied when we replace t with $m-1-1/q$, s with $1/q$, p with q', and n with $n-1$, so that

$$\|uv\|_{B^{1/q}_{q'q'}(\mathbb{S}^{n-1})} \leqslant C\|u\|_{B^{m-1-1/q}_{qq}(\mathbb{S}^{n-1})}\|v\|_{B^{1/q}_{q'q'}(\mathbb{S}^{n-1})}.$$

By dual, this implies that

$$\|uv\|_{B^{-1/q}_{qq}(\mathbb{S}^{n-1})} \leqslant C\|u\|_{B^{m-1-1/q}_{qq}(\mathbb{S}^{n-1})}\|v\|_{B^{-1/q}_{qq}(\mathbb{S}^{n-1})}.$$

Interpolating this inequality with (3.23) for $s > 0$, we see that (3.23) also holds for $-1/q \leqslant s \leqslant 0$ under the prescribed conditions. □

Lemma 3.4. *Let $m \geqslant 2$, $q \geqslant 1$ and $q > n/(m-1)$. Then for any $2 \leqslant k \leqslant m$ we have the following assertions*:

$$\mathcal{A} \in C^\infty\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), L\big(W^{k,q}\big(\mathbb{B}^n\big), W^{k-2,q}\big(\mathbb{B}^n\big)\big)\big), \tag{3.15}$$

$$\mathcal{D} \in C^\infty\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), L\big(W^{k,q}\big(\mathbb{B}^n\big), B_{qq}^{k-1-1/q}\big(\mathbb{S}^{n-1}\big)\big)\big), \tag{3.16}$$

$$\mathcal{P} \in C^\infty\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big) \times W^{m,q}\big(\mathbb{B}^n\big), L\big(W^{k,q}\big(\mathbb{B}^n\big), W^{k-1,q}\big(\mathbb{B}^n\big)\big)\big), \tag{3.17}$$

and for any $k > n/q$ we have

$$\mathcal{F}, \mathcal{G} \in C^\infty\big(W^{k,q}\big(\mathbb{B}^n\big), W^{k,q}\big(\mathbb{B}^n\big)\big). \tag{3.18}$$

Proof. (3.15) is an immediate consequence of Lemma 3.1 and the fact that $\Delta \in L(W^{k,q}(\Omega_\rho), W^{k-2,q}(\Omega_\rho))$ for any k. (3.16) is an immediate consequence of Lemmas 3.1, 3.2 and the fact that $\nabla \in L(W^{k,q}(\Omega_\rho), W^{k-1,q}(\Omega_\rho, \mathbb{R}^n))$ for any k. (3.17) follows from similar reasons as for (3.16). Finally, (3.18) follows from the fact that $W^{k,q}(\Omega_\rho)$ is an algebra under the condition $k > n/q$, as we mentioned earlier. □

Lemma 3.5. (i) *The mean curvature operator $\mathcal{K}(\rho)$ has the following splitting*:

$$\mathcal{K}(\rho) = \mathcal{L}(\rho)\rho + \mathcal{K}_1(\rho), \tag{3.19}$$

where $\mathcal{L}(\rho)$ is a second-order elliptic linear partial differential operator on $\mathbb{S}^{n-1}$, with coefficients being functions of ρ and its first-order derivatives, and $\mathcal{K}_1(\rho)$ is a first-order nonlinear partial differential operator on $\mathbb{S}^{n-1}$.

(ii) *Let $m \in \mathbb{N}$ and $q \geqslant 1$. Assume that $m \geqslant 4$, $q > n/(m-1)$ or $m = 3$, $n \geqslant 3$, $q > n/2$ or $m = 3$, $n < 3$, $2n/(n+1) \leqslant q < n/(n-1)$. Then we have*

$$\mathcal{L} \in C^\infty\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), L\big(B_{qq}^{k-1/q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{k-2-1/q}\big(\mathbb{S}^{n-1}\big)\big)\big), \quad 2 \leqslant k \leqslant m, \tag{3.20}$$

$$\mathcal{K}_1 \in C^\infty\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-1-1/q}\big(\mathbb{S}^{n-1}\big)\big), \tag{3.21}$$

so that

$$\mathcal{K} \in C^\infty\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-2-1/q}\big(\mathbb{S}^{n-1}\big)\big). \tag{3.22}$$

Proof. The assertion (i) is an immediate consequence of the mean curvature formula, see [20] and [21]. Next, since the condition $q > n/(m-1)$ implies that $B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1})$ is an algebra, (3.21) easily follows from the fact that $\mathcal{K}_1$ is a first-order nonlinear partial differential operator. Similarly, (3.20) follows from Corollary 3.3 and the facts that $B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1})$ is an algebra and $\mathcal{L}(\rho)$ is a second-order partial differential operator with coefficients being smooth functions of ρ and its first-order partial derivatives. Finally, (3.22) follows readily from (3.19)–(3.21). □

In order to perform the second step of reduction, we need the following lemma:

Lemma 3.6. *Let $m \geqslant 2$, $q \geqslant 1$, $q > n/(m-1)$ and $2 \leqslant k \leqslant m$. Given $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$ and $(w, \eta) \in W^{k-2,q}(\mathbb{B}^n) \times B_{qq}^{k-1/q}(\mathbb{S}^{n-1})$, the problem*

$$\begin{cases} -\mathcal{A}(\rho)u = w & \text{in } \mathbb{B}^n, \\ u = \eta & \text{on } \mathbb{S}^{n-1} \end{cases}$$

has a unique solution $u \in W^{k,q}(\mathbb{B}^n)$, and it has the following expression:

$$u = \mathcal{S}(\rho)w + \mathcal{T}(\rho)\eta,$$

where

$$\mathcal{S} \in C^\infty\big(\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}), L\big(W^{k-2,q}(\mathbb{B}^n), W^{k,q}(\mathbb{B}^n)\big)\big), \tag{3.23}$$

$$\mathcal{T} \in C^\infty\big(\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}), L\big(B_{qq}^{k-1/q}(\mathbb{S}^{n-1}), W^{k,q}(\mathbb{B}^n)\big)\big). \tag{3.24}$$

Proof. All assertions easily follow from the standard theory of elliptic partial differential equations, cf. the proof of Lemma 3.1 of [16]. □

In the sequel we perform the second step of reduction.

By Lemmas 3.5 and 3.6 we see that given $u \in W^{m-1,q}(\mathbb{B}^n)$ and $\rho \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$, the solution of Eq. (3.3) subject to the boundary value condition (3.5) is given by

$$v = \gamma\mathcal{T}(\rho)\mathcal{L}(\rho)\rho + \gamma\mathcal{T}(\rho)\mathcal{K}_1(\rho) + \mathcal{S}(\rho)\mathcal{G}(u).$$

Substitute this expression into (3.2) and (3.6), we see that the problem (3.2)–(3.8) is reduced into the following problem:

$$\partial_t u - c^{-1}\mathcal{A}(\rho)u - \mathcal{Q}(\rho, u)\rho = \mathcal{F}_1(\rho, u) \quad \text{in } \mathbb{B}^n \times (0, \infty), \tag{3.25}$$

$$\partial_t \rho - \mathcal{B}(\rho)\rho = \mathcal{G}_1(\rho, u) \quad \text{on } \mathbb{S}^{n-1} \times (0, \infty), \tag{3.26}$$

$$u = \bar{\sigma} \quad \text{on } \mathbb{S}^{n-1} \times (0, \infty), \tag{3.27}$$

$$u(0) = u_0 \quad \text{on } \mathbb{B}^n, \tag{3.28}$$

$$\rho(0) = \rho_0 \quad \text{on } \mathbb{S}^{n-1}, \tag{3.29}$$

where $\mathcal{A}(\rho)$ is as before, and

$$\begin{aligned}
\mathcal{B}(\rho)\zeta &= -\gamma\mathcal{D}(\rho)\mathcal{T}(\rho)\mathcal{L}(\rho)\zeta, \\
\mathcal{Q}(\rho, u)\zeta &= \mathcal{M}(\rho, u) \cdot \Pi\big(\mathcal{B}(\rho)\zeta\big), \\
\mathcal{F}_1(\rho, u) &= -c^{-1}\mathcal{F}(u) - \gamma\mathcal{P}(\rho, u)\mathcal{T}(\rho)\mathcal{K}_1(\rho) - \mathcal{P}(\rho, u)\mathcal{S}(\rho)\mathcal{G}(u) \\
&= -c^{-1}\mathcal{F}(u) - \mathcal{M}(\rho, u) \cdot \Pi\big(\mathcal{G}_1(\rho, u)\big), \\
\mathcal{G}_1(\rho, u) &= -\gamma\mathcal{D}(\rho)\mathcal{T}(\rho)\mathcal{K}_1(\rho) - \mathcal{D}(\rho)\mathcal{S}(\rho)\mathcal{G}(u).
\end{aligned}$$

To homogenize the boundary condition (3.27) we define

$$\mathcal{C}(\rho, u) = \mathcal{Q}(\rho, u + \bar{\sigma}), \qquad \mathcal{F}_2(\rho, u) = \mathcal{F}_1(\rho, u + \bar{\sigma}), \qquad \mathcal{G}_2(\rho, u) = \mathcal{G}_1(\rho, u + \bar{\sigma}).$$

Replacing $\mathcal{Q}$, $\mathcal{F}_1$ and $\mathcal{G}_1$ in (3.25) and (3.26) with $\mathcal{C}$, $\mathcal{F}_2$ and $\mathcal{G}_2$, respectively, we see that the inhomogeneous boundary value condition (3.27) is replaced by the homogeneous boundary value condition

$$u = 0 \quad \text{on } \mathbb{S}^{n-1} \times (0, \infty). \tag{3.30}$$

In what follows we assume that the conditions for m and q in the assertion (ii) of Lemma 3.5 are satisfied. We denote

$$U=\begin{pmatrix} u \\ \rho \end{pmatrix},\qquad \mathbb{A}(U)=\begin{pmatrix} c^{-1}\mathcal{A}(\rho) & \mathcal{C}(\rho,u) \\ 0 & \mathcal{B}(\rho) \end{pmatrix},\qquad \mathbb{F}_0(U)=\begin{pmatrix} \mathcal{F}_2(\rho,u) \\ \mathcal{G}_2(\rho,u) \end{pmatrix},\qquad U_0=\begin{pmatrix} \sigma_0-\bar{\sigma} \\ \rho_0 \end{pmatrix},$$

and

$$\mathbb{F}(U)=\mathbb{A}(U)U+\mathbb{F}_0(U).$$

We also denote

$$X=W^{m-3,q}\big(\mathbb{B}^n\big)\times B_{qq}^{m-3-1/q}\big(\mathbb{S}^{n-1}\big),\qquad X_0=\big(W^{m-1,q}\big(\mathbb{B}^n\big)\cap W_0^{1,q}\big(\mathbb{B}^n\big)\big)\times B_{qq}^{m-1/q}\big(\mathbb{S}^{n-1}\big),$$
$$Y=W^{m-2,q}\big(\mathbb{B}^n\big)\times B_{qq}^{m-2-1/q}\big(\mathbb{S}^{n-1}\big),$$

and

$$\mathcal{O}=\big(W^{m-1,q}\big(\mathbb{B}^n\big)\cap W_0^{1,q}\big(\mathbb{B}^n\big)\big)\times \mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big).$$

Then Eqs. (3.25), (3.26) (with $\mathcal{Q}$, $\mathcal{F}_1$, $\mathcal{G}_1$ respectively replaced with $\mathcal{C}$, $\mathcal{F}_2$, $\mathcal{G}_2$) and (3.30) are reduced into the following abstract differential equation in the Banach space X:

$$\frac{dU}{dt}=\mathbb{F}(U), \tag{3.31}$$

and the problem (3.25)–(3.29) is reduced into the following initial value problem:

$$\begin{cases} U'(t)=\mathbb{F}\big(U(t)\big) & \text{for } t>0, \\ U(0)=U_0. \end{cases} \tag{3.32}$$

Clearly, X, X_0 and Y are Banach spaces, $X_0 \hookrightarrow X$, Y is an intermediate space between X and X_0, and $\mathcal{O}$ is an open subset of X_0. From (3.15)–(3.18) and (3.20)–(3.24) we see that

$$\mathbb{A}\in C^\infty\big(\mathcal{O},L(X_0,X)\big),\qquad \mathbb{F}_0\in C^\infty(\mathcal{O},Y)\subseteq C^\infty(\mathcal{O},X),$$

so that $\mathbb{F}\in C^\infty(\mathcal{O},X)$. We note that for $m=3$, X_0 is dense in X, while for $m\geqslant 4$ the closure of X_0 in X is given by

$$\overline{X}_0=\big(W^{m-3,q}\big(\mathbb{B}^n\big)\cap W_0^{1,q}\big(\mathbb{B}^n\big)\big)\times B_{qq}^{m-3-1/q}\big(\mathbb{S}^{n-1}\big).$$

4. The Lie group action

For $\varepsilon>0$ we denote by $\mathbb{B}_\varepsilon^n$ the ball in $\mathbb{R}^n$ centered at the origin with radius ε. Regarding $\mathbb{B}_\varepsilon^n$ as a neighborhood of the unit element 0 of the commutative Lie group $\mathbb{R}^n$, we see that $G=\mathbb{B}_\varepsilon^n$ is a local Lie group of dimension n. In this section we introduce an action $\mathbf{S}^*$ of this (local) Lie group G to some open subset $\mathcal{O}'$ of X, $\mathcal{O}'\cap X_0=\mathcal{O}$, such that the relation

$$\mathbb{F}\big(\mathbf{S}_z^*(u)\big)=D\mathbf{S}_z^*(u)\mathbb{F}(u),\quad z\in G,\ u\in\mathcal{O}, \tag{4.1}$$

is satisfied.

Given $z\in\mathbb{R}^n$, we denote by S_z the translation in $\mathbb{R}^n$ induced by z, i.e.,

$$S_z(x)=x+z\quad \text{for } x\in\mathbb{R}^n.$$

Let $\rho \in C^1(\mathbb{S}^{n-1})$ such that $\|\rho\|_{C^1(\mathbb{S}^{n-1})}$ is sufficiently small, say, $\|\rho\|_{C^1(\mathbb{S}^{n-1})} < \delta$ for some small $\delta > 0$. For any $z \in \mathbb{B}^n_\varepsilon$, where ε is sufficiently small, consider the image of the hypersurface $r = 1 + \rho(\omega)$ under the translation S_z, which is still a hypersurface. This hypersurface has the equation $r = 1 + \tilde{\rho}(\omega)$ with $\tilde{\rho} \in C^1(\mathbb{S}^{n-1})$, and $\tilde{\rho}$ is uniquely determined by ρ and z. We denote

$$\tilde{\rho} = S_z^*(\rho).$$

Let $r_0 = |z|$ and $\omega_0 = z/|z|$. Then the explicit expression of $\tilde{\rho}$ is as follows:

$$\tilde{\rho}(\omega') = \sqrt{\big[1 + \rho(\omega)\big]^2 + r_0^2 + 2r_0\big[1 + \rho(\omega)\big]\omega \cdot \omega_0} - 1, \tag{4.2}$$

where $\omega' \in \mathbb{S}^{n-1}$ and $\omega \in \mathbb{S}^{n-1}$ are connected by the following relation:

$$\omega' = \frac{[1 + \rho(\omega)]\omega + r_0\omega_0}{\sqrt{[1 + \rho(\omega)]^2 + r_0^2 + 2r_0[1 + \rho(\omega)]\omega \cdot \omega_0}}. \tag{4.3}$$

In the sequel, the notations $\mathcal{O}_\delta(\mathbb{S}^{n-1})$ and $\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$ have the same meaning as in the previous section.

Lemma 4.1. *If ε and δ are sufficiently small then for any $z \in \mathbb{B}^n_\varepsilon$ and $\rho \in \mathcal{O}_\delta(\mathbb{S}^{n-1})$, $S_z^*(\rho)$ is well defined, and*

$$S_z^* \in C\big(\mathcal{O}_\delta\big(\mathbb{S}^{n-1}\big), C^1\big(\mathbb{S}^{n-1}\big)\big) \cap C^1\big(\mathcal{O}_\delta\big(\mathbb{S}^{n-1}\big), C\big(\mathbb{S}^{n-1}\big)\big).$$

Proof. Let $f_z(\rho, \omega)$ be the expression in the right-hand side of (4.3). We first prove that if ε is sufficiently small then for any $z \in \mathbb{B}^n_\varepsilon$ the mapping $\omega \to \omega' = f_z(\rho, \omega)$ from $\mathbb{S}^{n-1}$ to itself is an injection. Assume that $f_z(\rho, \omega_1) = f_z(\rho, \omega_2)$ for some $\omega_1, \omega_2 \in \mathbb{S}^{n-1}$. Then there exists $\lambda > 0$ such that

$$\big[1 + \rho(\omega_2)\big]\omega_2 + r_0\omega_0 = \lambda\big\{\big[1 + \rho(\omega_1)\big]\omega_1 + r_0\omega_0\big\}. \tag{4.4}$$

Let $\lambda = 1 + \mu$, $\omega_2 = \omega_1 + \xi$ and $\rho(\omega_2) = \rho(\omega_1) + \eta$, where $\mu \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$. Substituting these expressions into (4.4) we get

$$\big[1 + \rho(\omega_1)\big]\xi + \omega_2\eta = \mu\big\{\big[1 + \rho(\omega_1)\big]\omega_1 + r_0\omega_0\big\},$$

which yields $\xi = \mu\omega_1 + \zeta$, where $\zeta = (\mu r_0\omega_0 - \omega_2\eta)/[1 + \rho(\omega_1)]$. Since $|\rho(\omega_1)| < \delta$ and $|r_0| < \varepsilon$, from the expression of ζ we see that $|\zeta| \leqslant 2(\varepsilon|\mu| + |\eta|)$ if $\delta \leqslant 1/2$. Since $\max_{\omega \in \mathbb{S}^{n-1}} |\nabla_\omega \rho(\omega)| < \delta$, by the mean value theorem we easily deduce that $|\eta| \leqslant \delta|\xi|$, so that

$$|\zeta| \leqslant 2\big(\varepsilon|\mu| + \delta|\xi|\big). \tag{4.5}$$

From the relation $\xi = \mu\omega_1 + \zeta$ we have

$$|\xi| \leqslant |\mu| + |\zeta|. \tag{4.6}$$

Substituting the relation $\xi = \mu\omega_1 + \zeta$ into $\omega_2 = \omega_1 + \xi$ we get $\omega_2 = (1 + \mu)\omega_1 + \zeta$, or $(1 + \mu)\omega_1 = \omega_2 - \zeta$. From this relation and the fact that $|\mu| < 1$ (for ε and δ sufficiently small) we obtain

$$|\mu| \leqslant |\zeta|. \tag{4.7}$$

From (4.5)–(4.7) we can easily deduce that $|\zeta| = |\xi| = |\mu| = 0$ for sufficiently small ε and δ, which proves the desired assertion.

Next we prove that if ε and δ are sufficiently small then $D_\omega f_z(\rho,\omega): T_\omega(\mathbb{S}^{n-1}) \to T_\omega(\mathbb{S}^{n-1})$ is non-degenerate for any $\omega \in \mathbb{S}^{n-1}$ and $\rho \in \mathcal{O}_\delta(\mathbb{S}^{n-1})$. Note that since $\rho \in C^1(\mathbb{S}^{n-1})$, we have $f_z(\rho,\cdot) \in C^1(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$. Let $\mathbf{a} = [1+\rho(\omega)]\omega + r_0\omega_0$ and $\mathbf{b} = [1+\rho(\omega)]\xi + [\nabla\rho(\omega)\cdot\xi]\omega$, where $\xi \in T_\omega(\mathbb{S}^{n-1})$. Then a simple calculation shows that for any $\xi \in T_\omega(\mathbb{S}^{n-1})$ we have

$$D_\omega f_z(\rho,\omega)\xi = \frac{|\mathbf{a}|^2\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{a}}{|\mathbf{a}|^3}.$$

Since $\mathbf{a} = \omega + O(\delta+\varepsilon)$, $\mathbf{b} = \xi + O(\delta)|\xi|$ and $\omega\cdot\xi = 0$, from the above expression we see that $D_\omega f_z(\rho,\omega)\xi = \xi + O(\delta+\varepsilon)|\xi|$, so that the desired assertion holds.

It follows that for any $\rho \in \mathcal{O}_\delta(\mathbb{S}^{n-1})$ and $z \in \mathbb{B}^n_\varepsilon$, the mapping $f_z(\rho,\cdot): \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is open. As a result, $\operatorname{Im} f_z(\rho,\cdot)$ is an open subset of $\mathbb{S}^{n-1}$. Since $f_z(\rho,\cdot)$ is continuous, $\operatorname{Im} f_z(\rho,\cdot)$ is also closed in $\mathbb{S}^{n-1}$. Thus, $f_z(\rho,\cdot): \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ must be a surjection.

Now let $g_z(\rho,\cdot)$ be the inverse of $f_z(\rho,\cdot)$. By the inverse function theorem we know that $g_z(\rho,\cdot) \in C^1(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$. Let $F_z(\rho,\omega)$ denote the right-hand side of (4.2). Substituting $\omega = g_z(\rho,\omega')$ into (4.2) we see that

$$\tilde\rho(\omega') = F_z\big(\rho, g_z(\rho,\omega')\big) \quad \text{for } \omega' \in \mathbb{S}^{n-1}. \tag{4.8}$$

Hence, the mapping S_z^* is well defined, and $S_z^*(\rho) = F_z(\rho, g_z(\rho,\cdot))$.

Finally, it is clear that $F_z \in C^1(\mathcal{O}_\delta(\mathbb{S}^{n-1}) \times \mathbb{S}^{n-1}, \mathbb{R})$ and $f_z \in C^1(\mathcal{O}_\delta(\mathbb{S}^{n-1}) \times \mathbb{S}^{n-1}, \mathbb{S}^{n-1})$. By the implicit function theorem, we also have $g_z \in C^1(\mathcal{O}_\delta(\mathbb{S}^{n-1}) \times \mathbb{S}^{n-1}, \mathbb{S}^{n-1})$. Thus the mapping $(\rho,\omega) \to S_z^*(\rho)(\omega)$ from $\mathcal{O}_\delta(\mathbb{S}^{n-1}) \times \mathbb{S}^{n-1}$ to $\mathbb{R}$ is of C^1 class. Hence, we have $S_z^* \in C(\mathcal{O}_\delta(\mathbb{S}^{n-1}), C^1(\mathbb{S}^{n-1})) \cap C^1(\mathcal{O}_\delta(\mathbb{S}^{n-1}), C(\mathbb{S}^{n-1}))$. This completes the proof. $\square$

From the proof of Lemma 4.1 we can see that if $\rho \in \mathcal{O}^m_\delta(\mathbb{S}^{n-1}) = C^m(\mathbb{S}^{n-1}) \cap \mathcal{O}_\delta(\mathbb{S}^{n-1})$ for some $m \geqslant 2$, then F_z and f_z are of C^m class, which implies that g_z and the mapping $(\rho,\omega) \to S_z^*(\rho)(\omega) = F_z(\rho, g_z(\rho,\omega))$ are of C^m class, so that $S_z^* \in C^k(\mathcal{O}^m_\delta(\mathbb{S}^{n-1}), C^{m-k}(\mathbb{S}^{n-1}))$ for any $0 \leqslant k \leqslant m$. This particularly implies that

$$S_z^* \in C^\infty\big(C^\infty\big(\mathbb{S}^{n-1}\big) \cap \mathcal{O}_\delta\big(\mathbb{S}^{n-1}\big), C^\infty\big(\mathbb{S}^{n-1}\big)\big). \tag{4.9}$$

Similarly, if $\rho \in \mathcal{O}^{m+\mu}_\delta(\mathbb{S}^{n-1}) = C^{m+\mu}(\mathbb{S}^{n-1}) \cap \mathcal{O}_\delta(\mathbb{S}^{n-1})$ for some $m \geqslant 1$ and $0 < \mu \leqslant 1$, then $S_z^* \in C^k(\mathcal{O}^{m+\mu}_\delta(\mathbb{S}^{n-1}), C^{m-k+\mu}(\mathbb{S}^{n-1}))$ for any $0 \leqslant k \leqslant m$. To establish a similar result for the space $B^{m-1/q}_{qq}(\mathbb{S}^{n-1})$, we need the following lemma:

Lemma 4.2. *Let Ω_1, Ω_2 be two bounded smooth open subsets of $\mathbb{R}^n$. Let $m \geqslant 2$ and $q > n/(m-1)$. Let Φ be a diffeomorphism from $\overline{\Omega}_1$ to $\overline{\Omega}_2$. Assume that $\Phi \in W^{m,q}(\Omega_1,\mathbb{R}^n)$. Then $\Phi^{-1} \in W^{m,q}(\Omega_2,\mathbb{R}^n)$. Moreover, given $\varepsilon > 0$, the mapping $\Phi \to \Phi^{-1}$ from the set*

$$\big\{\Phi \in W^{m,q}\big(\Omega_1,\mathbb{R}^n\big): \big|\det D\Phi(x)\big| \geqslant \varepsilon \text{ for any } x \in \Omega_1\big\}$$

to $W^{m,q}(\Omega_2,\mathbb{R}^n)$ is C^∞, and there exists a continuous function $C_\varepsilon: [0,\infty) \to [0,\infty)$ such that

$$\big\|\Phi^{-1}\big\|_{W^{m,q}(\Omega_2,\mathbb{R}^n)} \leqslant C_\varepsilon\big(\|\Phi\|_{W^{m,q}(\Omega_1,\mathbb{R}^n)}\big). \tag{4.10}$$

Proof. We first note that the assumptions on m and q imply that $W^{m,q}(\Omega_1,\mathbb{R}^n) \hookrightarrow C^1(\overline{\Omega}_1,\mathbb{R}^n)$, and there exists constant $C > 0$ such that $\|\Phi\|_{C^1(\overline{\Omega}_1,\mathbb{R}^n)} \leqslant C\|\Phi\|_{W^{m,q}(\Omega_1,\mathbb{R}^n)}$. In the sequel we denote by x the variable in Ω_1, and by y the variable in Ω_2. We also denote $\Psi = \Phi^{-1}$. Then we have

$$D\Psi(y) = \big[D\Phi(x)\big]^{-1} = \big[\det D\Phi(x)\big]^{-1} D^*\Phi(x),$$

where $D^*\Phi(x)$ denotes the co-matrix of the matrix $D\Phi(x)$. By this formula, the Leibnitz rule and the Gagliardo–Nirenberg inequality we can easily deduce that for any $\alpha \in \mathbb{Z}_+^n$ such that $0 < |\alpha| \leqslant m$ and any $\varepsilon > 0$ such that $|\det D\Phi(x)| \geqslant \varepsilon$ for all $x \in \Omega_1$, we have

$$\left\|\partial^\alpha \Psi\right\|_{L^q(\Omega_2,\mathbb{R}^n)} \leqslant C\varepsilon^{-|\alpha|}\|D\Phi\|_{L^\infty(\Omega_1,\mathbb{R}^n)}^{n|\alpha|-2} \sum_{|\beta|=|\alpha|}\left\|\partial^\beta \Phi\right\|_{L^q(\Omega_1,\mathbb{R}^n)} \leqslant C\varepsilon^{-|\alpha|}\|\Phi\|_{W^{m,q}(\Omega_1,\mathbb{R}^n)}^{n|\alpha|-1}.$$

Hence (4.10) holds. The assertion that the mapping $\Phi \to \Phi^{-1}$ is smooth is an immediate consequence of the above argument. $\square$

Lemma 4.3. *Let m and q be as in Lemma 4.2. Then we have the following assertions*:

(i) *For $\delta > 0$ sufficiently small and for $z \in \mathbb{B}_\varepsilon^n$ with ε sufficiently small, we have*

$$S_z^* \in C^k\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-k-1/q}\big(\mathbb{S}^{n-1}\big)\big), \quad 0 \leqslant k \leqslant m-1.$$

In particular, $S_z^* \in C(\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}), B_{qq}^{m-1/q}(\mathbb{S}^{n-1})) \cap C^1(\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}), B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1}))$. Moreover, for any $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$ and $1 \leqslant k \leqslant m-1$, the operator $DS_z^*(\rho)$ from $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ to $B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1})$ can be uniquely extended to the space $B_{qq}^{m-k-1/q}(\mathbb{S}^{n-1})$, such that

$$DS_z^*(\rho) \in L\big(B_{qq}^{m-k-1/q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-k-1/q}\big(\mathbb{S}^{n-1}\big)\big),$$

and the mapping $\rho \to DS_z^*(\rho)$ from $\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$ into $L(B_{qq}^{m-k-1/q}(\mathbb{S}^{n-1}), B_{qq}^{m-k-1/q}(\mathbb{S}^{n-1}))$ is continuous.

(ii) *For any $z, w \in \mathbb{B}_\varepsilon^n$ with ε sufficiently small, we have*

$$S_z^* \circ S_w^* = S_{z+w}^*, \quad S_0^* = \mathrm{id}, \quad \textit{and} \quad \big(S_z^*\big)^{-1} = S_{-z}^*.$$

(iii) *The mapping $S^* : z \to S_z^*$ from $\mathbb{B}_\varepsilon^n$ to $C(\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}), B_{qq}^{m-1/q}(\mathbb{S}^{n-1}))$ is an injection, and*

$$S^* \in C^k\big(\mathbb{B}_\varepsilon^n, C^l\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-k-l-1/q}\big(\mathbb{S}^{n-1}\big)\big)\big), \quad k \geqslant 0,\ l \geqslant 0,\ k+l \leqslant m-1.$$

(iv) *Finally assume that $2 \leqslant k < m$, $q > n/(k-1)$ and define $p : \mathbb{B}_\varepsilon^n \times \mathcal{O}_\delta^{k,q}(\mathbb{S}^{n-1}) \to B_{qq}^{k-1/q}(\mathbb{S}^{n-1})$ by $p(z,\rho) = S_z^*(\rho)$. Then for any $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$ the mapping $z \to p(z,\rho)$ from $\mathbb{B}_\varepsilon^n$ to $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ is Fréchet differentiable when regarded as a mapping from $\mathbb{B}_\varepsilon^n$ to $B_{qq}^{k-1/q}(\mathbb{S}^{n-1})$, and we have $\operatorname{rank} D_z p(z,\rho) = n$ for any $z \in \mathbb{B}_\varepsilon^n$ and $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$. If furthermore $\rho \in C^\infty(\mathbb{S}^{n-1})$ then $[z \to p(z,\rho)] \in C^\infty(\mathbb{B}_\varepsilon^n, C^\infty(\mathbb{S}^{n-1})) \subseteq C^\infty(\mathbb{B}_\varepsilon^n, B_{qq}^{m-1/q}(\mathbb{S}^{n-1}))$.*

Proof. We first note that the assumptions on m and q imply that $B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \hookrightarrow C^1(\mathbb{S}^{n-1})$, so that by Lemma 4.1, $S_z^*(\rho)$ makes sense for $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$. Next, by (4.8) we see that $S_z^*(\rho) = F_z(\rho, g_z(\rho,\cdot))$. Considering (4.2) and (4.3), for given $z = r_0\omega_0 \in \mathbb{B}_\varepsilon^n$ and any $u \in W^{m,q}(\mathbb{B}^n)$ such that $\|u\|_{C^1(\mathbb{B}^n)} < \delta$ we define

$$\tilde{u}(x') = \sqrt{\big[1+u(x)\big]^2 + r_0^2 + 2r_0\big[1+u(x)\big]x\cdot\omega_0} - 1, \quad x \in \overline{\mathbb{B}}^n, \tag{4.11}$$

where x' and x are related by

$$x' = \frac{[1+u(x)]x + r_0\omega_0}{|[1+u(x)]x + r_0\omega_0|}|x|\phi\big(|x|-1\big) + \big[1-\phi\big(|x|-1\big)\big]x, \quad x \in \overline{\mathbb{B}}^n, \tag{4.12}$$

where ϕ is as in Section 3. As before we use the notation $F_z(u,x)$ to denote the expression on the right-hand side of (4.11). Since the assumptions on m and q imply that $W^{m,q}(\mathbb{B}^n)$ is an algebra, it is clear that $F_z(u,\cdot) \in W^{m,q}(\mathbb{B}^n)$, and the mapping $u \to F_z(u,\cdot)$ is C^∞. We also use the same notation $f_z(u,x)$ as before to denote the expression on the right-hand side of (4.12), because if we particularly take $u = \Pi(\rho)$ and $x = \omega \in \mathbb{S}^{n-1}$ then we get $f_z(\rho,\omega)$ defined before. It can be easily shown that if ε and δ are sufficiently small then the mapping $\Phi_u : x \to x' = f_z(u,x)$ is a diffeomorphism of $\bar{\mathbb{B}}^n$ to itself and $\det D\Phi_u(x) = 1 + O(\varepsilon + \delta)$. Moreover, since $W^{m,q}(\mathbb{B}^n)$ is an algebra, we have $\Phi_u \in W^{m,q}(\mathbb{B}^n, \mathbb{R}^n)$ and it is clear that the mapping $u \to \Phi_u$ is C^∞. By Lemma 4.2 we infer that $\Phi_u^{-1} \in W^{m,q}(\mathbb{B}^n, \mathbb{R}^n)$, and the mapping $\Phi_u \to \Phi_u^{-1}$ is C^∞. Substituting $x = \Phi_u^{-1}(x')$ into the right-hand side of (4.11) and using Lemma 3.1, we see that $\tilde{u} = F_z(u, \Phi_u^{-1}(\cdot)) \in W^{m,q}(\mathbb{B}^n)$. Now, clearly if $u = \Pi(\rho)$ for some $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$ then we have $\tilde{u}|_{\mathbb{S}^{n-1}} = S_z^*(\rho)$, so that we have proved that $S_z^*(\rho) \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ for any $\rho \in \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$. We note that though both the mappings $u \to F_z(u,\cdot)$ and $u \to \Phi_u^{-1}$ are C^∞, the mapping $u \to \tilde{u} = F_z(u, \Phi_u^{-1}(\cdot))$ is, however, not necessarily C^∞, because $F_z(u,x)$ is generally not C^∞ in x. Despite of this inconvenience, we still can ensure that the mapping $u \to \tilde{u} = F_z(u, \Phi_u^{-1}(\cdot))$ from $W^{m,q}(\mathbb{B}^n) \cap \{u \in C^1(\bar{\mathbb{B}}^n)\colon \|u\|_{C^1(\bar{\mathbb{B}}^n)} < \delta\}$ to $W^{m,q}(\mathbb{B}^n)$ is continuous, because both $(u,x) \to F_z(u,x)$ and $u \to \Phi_u^{-1}$ are continuous. Thus $S_z^* \in C(\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}), B_{qq}^{m-1/q}(\mathbb{S}^{n-1}))$. Next, since $S_z^*(\rho) = \Gamma F_z(\Pi(\rho), \Phi_{\Pi(\rho)}^{-1}(\cdot))$, where Γ denotes the trace operator, we have

$$\begin{aligned} DS_z^*(\rho)\eta &= \Gamma D_1 F_z\big(\Pi(\rho), \Phi_{\Pi(\rho)}^{-1}(\cdot)\big)\Pi(\eta) + \Gamma D_2 F_z\big(\Pi(\rho), \Phi_{\Pi(\rho)}^{-1}(\cdot)\big) D_u \Phi_{\Pi(\rho)}^{-1}(\cdot)\Pi(\eta) \\ &\equiv I(\rho)\eta + II(\rho)\eta, \end{aligned} \tag{4.13}$$

where D_1 and D_2 represent the Fréchet derivatives in the first and the second arguments, respectively, and $D_u\Phi_{\Pi(\rho)}^{-1} = D_u\Phi_u^{-1}|_{u=\Pi(\rho)}$. By Lemma 3.2(i) it is obvious that

$$\begin{aligned} I \in\, & C\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), L\big(B_{qq}^{m-1/q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-1/q}\big(\mathbb{S}^{n-1}\big)\big)\big) \\ & \cap C\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), L\big(B_{qq}^{m-k-1/q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-k-1/q}\big(\mathbb{S}^{n-1}\big)\big)\big), \quad 1 \leqslant k \leqslant m-1. \end{aligned}$$

To treat II we denote $G_z(t,y) = \sqrt{(1+t)^2 + r_0^2 + 2r_0(1+t)y\cdot\omega_0} - 1$ for $t \in \mathbb{R}$ and $y \in \mathbb{B}^n$. Then $F_z(u,x) = G_z(u(x),x)$ for $u \in W^{m,q}(\mathbb{B}^n)$ and $x \in \mathbb{B}^n$, so that

$$D_2F_z(u,x) = D_1G_z\big(u(x),x\big)Du(x) + D_2G_z\big(u(x),x\big).$$

Given $u \in W^{m,q}(\mathbb{B}^n)$, from the above expression of $D_2F_z(u,x)$ we see that $D_2F_z(u,\cdot) = [x \to D_2F_z(u,x)] \in W^{m-1,q}(\mathbb{B}^n, L(\mathbb{R}^n,\mathbb{R}))$. Besides, since

$$D_u\Phi_u^{-1} \in L\big(W^{m,q}\big(\mathbb{B}^n\big), W^{m,q}\big(\mathbb{B}^n,\mathbb{R}^n\big)\big) \cap L\big(W^{m-k,q}\big(\mathbb{B}^n\big), W^{m-k,q}\big(\mathbb{B}^n,\mathbb{R}^n\big)\big)$$

($1 \leqslant k \leqslant m-1$), for any $\eta \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ we have $D_u\Phi_u^{-1}(\cdot)\pi(\eta) = [x \to D_u\Phi_u^{-1}(x)\pi(\eta)] \in W^{m,q}(\mathbb{B}^n,\mathbb{R}^n)$, and if $\eta \in B_{qq}^{m-k-1/q}(\mathbb{S}^{n-1})$ for some $1 \leqslant k \leqslant m-1$ then we have $D_u\Phi_u^{-1}(\cdot)\pi(\eta) \in W^{m-k,q}(\mathbb{B}^n,\mathbb{R}^n)$. Hence, given $\rho \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$, for any $\eta \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ we have

$$D_2F_z\big(\pi(\rho), \Phi_{\pi(\rho)}^{-1}(\cdot)\big)D_u\Phi_{\pi(\rho)}^{-1}(\cdot)\pi(\eta) = \big[x \to D_2F_z\big(\pi(\rho), \Phi_{\pi(\rho)}^{-1}(x)\big)D_u\Phi_{\pi(\rho)}^{-1}(x)\pi(\eta)\big] \in W^{m-1,q}\big(\mathbb{B}^n\big),$$

and if $\eta \in B_{qq}^{m-k-1/q}(\mathbb{S}^{n-1})$ for some $1 \leqslant k \leqslant m-1$ then we have

$$D_2F_z\big(\pi(\rho), \Phi_{\pi(\rho)}^{-1}(\cdot)\big)D_u\Phi_{\pi(\rho)}^{-1}(\cdot)\pi(\eta) \in W^{m-k,q}\big(\mathbb{B}^n\big).$$

This implies that for $\rho \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$, if $\eta \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ then $II(\rho)\eta \in B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1})$, whereas if $\eta \in B_{qq}^{m-k-1/q}(\mathbb{S}^{n-1})$ for some $1 \leqslant k \leqslant m-1$ then $II(\rho)\eta \in B_{qq}^{m-k-1/q}(\mathbb{S}^{n-1})$. A similar analysis shows that

$$\begin{aligned} II \in\ & C\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), L\big(B_{qq}^{m-1/q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-1-1/q}\big(\mathbb{S}^{n-1}\big)\big)\big) \\ &\cap C\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), L\big(B_{qq}^{m-k-1/q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-k-1/q}\big(\mathbb{S}^{n-1}\big)\big)\big), \quad 1 \leqslant k \leqslant m-1. \end{aligned}$$

Hence, $S_z^* \in C^1(\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}), B_{qq}^{m-1-1/q}(\mathbb{S}^{n-1}))$, and

$$\begin{aligned} DS_z^* \in\ & C\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), L\big(B_{qq}^{m-1/q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-1-1/q}\big(\mathbb{S}^{n-1}\big)\big)\big) \\ &\cap C\big(\mathcal{O}_\delta^{m,q}\big(\mathbb{S}^{n-1}\big), L\big(B_{qq}^{m-k-1/q}\big(\mathbb{S}^{n-1}\big), B_{qq}^{m-k-1/q}\big(\mathbb{S}^{n-1}\big)\big)\big), \quad 1 \leqslant k \leqslant m-1. \end{aligned}$$

Furthermore, by an induction argument we see that $S_z^* \in C^k(\mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1}), B_{qq}^{m-k-1/q}(\mathbb{S}^{n-1}))$ for any $0 \leqslant k \leqslant m-1$. This proves assertion (i). Assertion (ii) is obvious. The first part of assertion (iii) is evident, and the second part follows by checking more carefully the argument in the proof of assertion (i), which we omit here. From the proof of assertion (i) we see that for any integers $2 \leqslant k < m$ and $q > n/(k-1)$, the mapping $p : \mathbb{B}_\varepsilon^n \times \mathcal{O}_\delta^{k,q}(\mathbb{S}^{n-1}) \to B_{qq}^{k-1/q}(\mathbb{S}^{n-1})$ defined by $p(z,\rho) = S_z^*(\rho)$ is continuously differentiable at any point $(z,\rho) \in \mathbb{B}_\varepsilon^n \times \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$. Moreover, a simple calculation shows that $D_1 p(0,0)z = z\cdot\omega$. Here $z\cdot\omega$ represents the function $\omega \to z\cdot\omega$ on $\mathbb{S}^{n-1}$. This shows that $\operatorname{rank} D_1 p(0,0) = n$. By continuity, we infer that $\operatorname{rank} D_1 p(z,\rho) = n$ for any $(z,\rho) \in \mathbb{B}_\varepsilon^n \times \mathcal{O}_\delta^{m,q}(\mathbb{S}^{n-1})$, provided ε and δ are sufficiently small. Finally, if $\rho \in C^\infty(\mathbb{S}^{n-1})$ then from the construction of S_z^* it is clear that $[z \to p(z,\rho)] \in C^\infty(\mathbb{B}_\varepsilon^n, C^\infty(\mathbb{S}^{n-1}))$. Hence, assertion (iv) follows. This completes the proof. □

By Lemma 4.3 we see that the mapping S^* provides an action of the local group $G = \mathbb{B}_\varepsilon^n$ to some open subset of $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$. We note that if $c = 0$ then by some similar arguments as in Section 3 we can reduce the problem (1.1)–(1.5) and (1.7) into a differential equation $\rho'(t) = \mathcal{A}_\gamma(\rho(t))$ in the Banach space $B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$ in the unknown function $\rho = \rho(t)$ only, where $\mathcal{A}_\gamma$ is defined in some open subset of $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$. It can be shown that in this case the reduced equation satisfies a similar relation as that in (4.1) under the above action of G (cf. the proof of Lemma 4.6 below). For Eq. (3.31), however, G has to act on some open set in X. This is fulfilled in the following paragraph. In the sequel, the notations X, X_0 and $\mathcal{O}$ have the same meaning as introduced in the end of Section 3.

Given $z \in \mathbb{B}_\varepsilon^n$ and $\rho \in \mathcal{O}_\delta(\mathbb{S}^{n-1})$, let $P_{z,\rho} : C(\mathbb{B}^n) \to C(\mathbb{B}^n)$ be the mapping

$$P_{z,\rho}(u)(x) = u\big(\Theta_\rho^{-1}\big(\Theta_{S_z^*(\rho)}(x) - z\big)\big) \quad \text{for } u \in C\big(\mathbb{B}^n\big).$$

Clearly, $P_{z,\rho} \in L(C(\mathbb{B}^n), C(\mathbb{B}^n))$. Moreover, if $\rho \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ then $S_z^*(\rho) \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$, so that by Lemma 3.1 we have $P_{z,\rho} \in L(W^{m,q}(\mathbb{B}^n), W^{m,q}(\mathbb{B}^n))$. For $u \in C(\mathbb{B}^n)$, $\rho \in \mathcal{O}_\delta(\mathbb{S}^{n-1})$ and $z \in \mathbb{B}_\varepsilon^n$ we denote

$$\mathbf{S}_z^* \begin{pmatrix} u \\ \rho \end{pmatrix} = \begin{pmatrix} P_{z,\rho}(u) \\ S_z^*(\rho) \end{pmatrix}.$$

Note that $\mathbf{S}_0^* = \mathrm{id}$.

Lemma 4.4. *Let $m \geqslant 5$ and $q > n/(m-4)$. Let*

$$\mathcal{O}' = W^{m-3}\big(\mathbb{B}^n\big) \times \big(B_{qq}^{m-3-1/q}\big(\mathbb{S}^{n-1}\big) \cap \mathcal{O}_\delta\big(\mathbb{S}^{n-1}\big)\big) \quad (\Rightarrow \mathcal{O} = X_0 \cap \mathcal{O}').$$

For sufficiently small $\varepsilon > 0$ and $\delta > 0$ we have the following assertions:

(i) *For any $\varepsilon \in \mathbb{B}^n_\varepsilon$ we have $\mathbf{S}^*_z \in C(\mathcal{O}', X) \cap C(\mathcal{O}, X_0)$. Moreover, regarded as a mapping from $\mathcal{O}'$ to X, $\mathbf{S}^*_z$ is Fréchet differentiable at every point in $\mathcal{O}$, and $D\mathbf{S}^*_z \in C(\mathcal{O}, L(X, X))$.*

(ii) *For any $z, w \in \mathbb{B}^n_\varepsilon$ we have*

$$\mathbf{S}^*_z \circ \mathbf{S}^*_w = \mathbf{S}^*_{z+w}, \quad \mathbf{S}^*_0 = \mathrm{id}, \quad \textit{and} \quad \big(\mathbf{S}^*_z\big)^{-1} = \mathbf{S}^*_{-z}.$$

(iii) *The mapping $\mathbf{S}^* : z \to \mathbf{S}^*_z$ from $\mathbb{B}^n_\varepsilon$ to $C(\mathcal{O}', X)$ is an injection, and*

$$\mathbf{S}^* \in C^k\big(\mathbb{B}^n_\varepsilon, C^l\big(\mathcal{O}, W^{m-k-l-1,q}\big(\mathbb{B}^n\big) \times B^{m-k-l-1/q}_{qq}\big(\mathbb{S}^{n-1}\big)\big)\big), \quad k \geqslant 0,\ l \geqslant 0,\ k+l \leqslant m-1. \tag{4.14}$$

Moreover, for fixed $z \in \mathbb{B}^n_\varepsilon$ we have

$$D\mathbf{S}^*_z \in C\big(\mathcal{O}, L(X)\big), \tag{4.15}$$

i.e., for any $U \in \mathcal{O}$, the operator $D\mathbf{S}^*_z(U)$ (which is, by taking $k = 0$ and $l = 1$ in (4.14), a bounded linear operator from $X_0 = (W^{m-1,q}(\mathbb{B}^n) \cap W^{1,q}_0(\mathbb{B}^n)) \times B^{m-1/q}_{qq}(\mathbb{S}^{n-1})$ to $W^{m-2,q}(\mathbb{B}^n) \times B^{m-1-1/q}_{qq}(\mathbb{S}^{n-1})$) can be extended into a bounded linear operator from $X = W^{m-3,q}(\mathbb{B}^n) \times B^{m-3-1/q}_{qq}(\mathbb{S}^{n-1})$ to itself, and the mapping $U \to D\mathbf{S}^*_z(U)$ from $\mathcal{O}$ to $L(X)$ is continuous.

(iv) *Define $p : \mathbb{B}^n_\varepsilon \times \mathcal{O}' \to X$ by $p(z, U) = \mathbf{S}^*_z(U)$. Then for any $U \in \mathcal{O}$ we have $p(\cdot, U) \in C^1(\mathbb{B}^n_\varepsilon, X)$, and $\operatorname{rank} D_z p(z, U) = n$ for every $z \in \mathbb{B}^n_\varepsilon$ and $U \in \mathcal{O}$. If furthermore $U \in X^\infty = C^\infty(\overline{\mathbb{B}}^n) \times C^\infty(\mathbb{S}^{n-1})$ then $p(\cdot, U) \in C^\infty(\mathbb{B}^n_\varepsilon, X^\infty)$.*

Proof. All assertions of this lemma follow readily from the corresponding assertions in Lemma 4.3. □

In the sequel, for $\rho = \rho(t)$, $u = u(x, t)$ and $U = \binom{u(x,t)}{\rho(t)}$, we denote by $P_{z,\rho}(u)$ the function $\tilde{u}(x, t) = u(\Theta^{-1}_{\rho(t)}(\Theta_{S^*_z(\rho(t))}(x) - z), t)$, by $S^*_z(\rho)$ the function $\tilde{\rho}(t) = S^*_z(\rho(t))$, and by $\mathbf{S}^*_z(U)$ the vector function $\binom{P_{z,\rho}(u)}{S^*_z(\rho)} = \binom{\tilde{u}(x,t)}{\tilde{\rho}(t)}$.

Lemma 4.5. *If $U = \binom{u}{\rho}$ is a solution of Eq. (3.31) such that $\|\rho\|_{C^1(\mathbb{S}^{n-1})}$ is sufficiently small, then for any $z \in \mathbb{R}^n$ such that $|z|$ is sufficiently small, $\mathbf{S}^*_z(U) = \binom{P_{z,\rho}(u)}{S^*_z(\rho)}$ is also a solution of (3.31).*

Proof. It is easy to see that if a triple (σ, p, Ω) is a solution of the problem (1.1)–(1.5), then for any $z \in \mathbb{R}^n$ the triple $(\tilde{\sigma}, \tilde{p}, \tilde{\Omega})$ defined by

$$\tilde{\sigma}(x, t) = \sigma(x - z, t), \qquad \tilde{p}(x, t) = p(x - z, t), \qquad \tilde{\Omega}(t) = \Omega(t) + z,$$

is also a solution of that problem. From this fact one can easily verify that if $U = \binom{u}{\rho}$ is a solution of Eq. (3.31) then $\tilde{U} = \binom{\tilde{u}}{\tilde{\rho}}$, where

$$\tilde{u}(x, t) = u\big(\Theta^{-1}_{\rho(t)}\big(\Theta_{S^*_z(\rho(t))}(x) - z\big), t\big), \qquad \tilde{\rho}(t) = S^*_z\big(\rho(t)\big),$$

is also a solution of that equation, which is the desired assertion. □

Lemma 4.6. *The following relation holds for any $z \in \mathbb{B}^n_\varepsilon$ and any $U = \binom{u}{\rho} \in \mathcal{O}$, provided ε and δ are sufficiently small:*

$$\mathbb{F}\big(\mathbf{S}^*_z(U)\big) = D\mathbf{S}^*_z(U)\mathbb{F}(U). \tag{4.16}$$

Proof. By Theorem 1.1 of [14], given any $U = \binom{u}{\rho} \in X_0$ there exists $\delta > 0$ such that Eq. (3.31) has a unique solution $V = V(t)$ for $0 \leqslant t \leqslant \delta$, which belongs to $C([0,\delta], X) \cap C((0,\delta], \mathcal{O}) \cap L^\infty((0,\delta), X_0) \cap C^1((0,\delta], X)$ and satisfies the initial condition $V(0) = U$. (This result also follows from Corollary 5.3 in the next section and a standard existence theorem that we used in the proof of Theorem 2.1.) Let $\tilde V(t) = \mathbf{S}_z^*(V(t))$ for $0 \leqslant t \leqslant \delta$. By Lemma 4.5, $\tilde V$ is also a solution of (3.31), satisfying the initial condition $\tilde V(0) = \mathbf{S}_z^*(U)$. The fact that $\tilde V$ is the solution of (3.31) implies that

$$\frac{d\tilde V(t)}{dt} = \mathbb{F}\big(\tilde V(t)\big) \quad \text{for } 0 < t \leqslant \delta.$$

On the other hand, since $\tilde V(t) = \mathbf{S}_z^*(V(t))$, we have

$$\frac{d\tilde V(t)}{dt} = D\mathbf{S}_z^*\big(V(t)\big)\frac{dV(t)}{dt} = D\mathbf{S}_z^*\big(V(t)\big)\mathbb{F}\big(V(t)\big) \quad \text{for } 0 < t \leqslant \delta.$$

Thus $\mathbb{F}(\tilde V(t)) = D\mathbf{S}_z^*(V(t))\mathbb{F}(V(t))$ for $0 < t \leqslant \delta$. If $V(t)$ is a strict solution then clearly $\tilde V(t)$ is also a strict solution, so that by directly letting $t \to 0^+$ we get (4.16). If $V(t)$ is not a strict solution then we appeal to the quasi-linear structure of $\mathbb{F}(U)$ to prove (4.16): Since $V \in L^\infty((0,\delta), X_0) \cap C([0,\delta], X)$ and $V(0) = U$, we infer that $V(t)$ weakly converges to U in X_0 as $t \to 0^+$. Similarly $\tilde V(t)$ weakly converges to $\mathbf{S}_z^*(U)$ in X_0. Since $\mathbb{F}(U) = \mathbb{A}(U)U + \mathbb{F}_0(U)$, we have

$$\begin{aligned}\mathbb{F}\big(V(t)\big) - \mathbb{F}(U) &= \big[\mathbb{A}\big(V(t)\big) - \mathbb{A}(U)\big]V(t) + \mathbb{A}(U)\big[V(t) - U\big] + \big[\mathbb{F}_0\big(V(t)\big) - \mathbb{F}_0(U)\big] \\ &\equiv I(t) + II(t) + III(t).\end{aligned}$$

We have $\|I(t)\|_X \leqslant C\|\mathbb{A}(V(t)) - \mathbb{A}(U)\|_{L(X_0,X)}$, so that $\lim_{t\to 0^+}\|I(t)\|_X = 0$, because $\mathbb{A}$ maps X_0 compactly into $L(X_0, X)$. We also have $\lim_{t\to 0^+}\|III(t)\|_X = 0$ by a similar reason. In addition, it is evident that $II(t)$ weakly converges to 0 in X as $t \to 0^+$. Therefore, $\mathbb{F}(V(t))$ weakly converges to $\mathbb{F}(U)$ in X. Similarly, $\mathbb{F}(\tilde V(t))$ weakly converges to $\mathbb{F}(\mathbf{S}_z^*(U))$ in X. Finally, from the expression of $D\mathbf{S}_z^*$ (cf. (4.13)) we can easily find that $D\mathbf{S}_z^*$ maps X_0 compactly into $L(X, X)$. Thus by a similar argument as above we infer that $D\mathbf{S}_z^*(V(t))\mathbb{F}(V(t))$ weakly converges to $D\mathbf{S}_z^*(U)\mathbb{F}(U)$ in X as $t \to 0^+$. Hence (4.16) holds. $\square$

Lemma 4.6 has some obvious corollaries. First, let $\mathbb{F}_2$ be the second component of $\mathbb{F}$. Taking the second components of both sides of (4.16) we get

$$\mathbb{F}_2\big(\mathbf{S}_z^*(U)\big) = DS_z^*(\rho)\mathbb{F}_2(U), \tag{4.17}$$

where ρ is the second component of U. Next, let $u_s = \sigma_s - \bar\sigma$ and $U_s = \binom{u_s}{0}$. U_s is a stationary point of Eq. (3.31), i.e., $\mathbb{F}(U_s) = 0$. Taking $U = U_s$ in (4.16) we get $\mathbb{F}(\mathbf{S}_z^*(U_s)) = 0$ for any $z \in \mathbb{B}_\varepsilon^n$. Since clearly $U_s \in X^\infty$, we have $[z \to \mathbf{S}_z^*(U_s)] \in C^\infty(\mathbb{B}_\varepsilon^n, X^\infty)$. Thus, differentiating the equation $\mathbb{F}(\mathbf{S}_z^*(U_s)) = 0$ in z at $z = 0$, we obtain

$$\mathbb{F}'(U_s)W_j = 0, \qquad W_j = \begin{pmatrix} [\phi(r-1)r-1]\sigma_s'(r)\omega_j \\ \omega_j \end{pmatrix}, \quad j = 1, 2, \ldots, n, \tag{4.18}$$

i.e., 0 is an eigenvalue of $\mathbb{F}'(U_s)$ and $W_1, W_2, \ldots, W_n$ are corresponding eigenvectors. Later we shall see that the multiplicity of 0 is exactly n (see Corollary 6.4).

5. Calculation of $\mathbb{F}'(U_s)$

In this section we calculate the Fréchet derivative of $\mathbb{F}$ at the stationary point U_s. Since $\mathbb{F}(U) = \mathbb{A}(U)U + \mathbb{F}_0(U)$, we have

$$\mathbb{F}'(U_s)V = \mathbb{A}(U_s)V + \big[\mathbb{A}'(U_s)V\big]U_s + \mathbb{F}_0'(U_s)V \quad \text{for } V \in X_0. \tag{5.1}$$

Recall that $\mathbb{A} \in C^\infty(\mathcal{O}, L(X_0, X))$, so that $\mathbb{A}'(U_s) \in L(X_0, L(X_0, X))$, and $\mathbb{A}'(U_s)V \in L(X_0, X)$ for $V \in X_0$. Since $U_s = \binom{u_s}{0}$, simple calculations show that for any $V = \binom{v}{\eta} \in X_0$ we have

$$\mathbb{A}(U_s)V = \begin{pmatrix} c^{-1}\mathcal{A}(0)v + \mathcal{C}(0, u_s)\eta \\ \mathcal{B}(0)\eta \end{pmatrix}, \qquad \big[\mathbb{A}'(U_s)V\big]U_s = \begin{pmatrix} c^{-1}[\mathcal{A}'(0)\eta]u_s \\ 0 \end{pmatrix}, \tag{5.2}$$

and

$$\mathbb{F}_0'(U_s)V = \begin{pmatrix} D_u\mathcal{F}_2(0, u_s)v + D_\rho\mathcal{F}_2(0, u_s)\eta \\ D_u\mathcal{G}_2(0, u_s)v + D_\rho\mathcal{G}_2(0, u_s)\eta \end{pmatrix}, \tag{5.3}$$

where $D_u\mathcal{F}_2$ and $D_\rho\mathcal{F}_2$ represent Fréchet derivatives of $\mathcal{F}_2(\rho, u)$ in u and ρ, respectively, and similarly for $D_u\mathcal{G}_2$ and $D_\rho\mathcal{G}_2$. Clearly,

$$\mathcal{A}(0)v = \Delta v, \tag{5.4}$$

and a simple computation shows that

$$\mathcal{C}(0, u_s)\eta = \phi(r-1)\sigma_s'(r)\Pi\big(\mathcal{B}(0)\eta\big). \tag{5.5}$$

To compute $\mathcal{B}(0)\eta = -\gamma\mathcal{D}(0)\mathcal{T}(0)\mathcal{L}(0)\eta$ we first note that, clearly,

$$\mathcal{D}(0)v = \frac{\partial v}{\partial r}\bigg|_{r=1} \quad \text{and} \quad \mathcal{T}(0)\eta = \Pi(\eta).$$

Next, recall that

$$\mathcal{K}(\rho) = \mathcal{L}(\rho)\rho + \mathcal{K}_1(\rho), \quad \text{so that} \quad \mathcal{K}'(0)\eta = \mathcal{L}(0)\eta + \mathcal{K}_1'(0)\eta. \tag{5.6}$$

On the other hand, from [25] we know that

$$\mathcal{K}(\varepsilon\eta) = 1 - \varepsilon\left[\eta(\omega) + \frac{1}{n-1}\Delta_\omega\eta(\omega)\right] + o(\varepsilon),$$

which implies that $\mathcal{K}(0) = 1$ and $\mathcal{K}'(0)\eta = -[\eta + \frac{1}{n-1}\Delta_\omega\eta]$. Comparing these expressions with those in (5.6), we obtain

$$\mathcal{L}(0)\eta = -\frac{1}{n-1}\Delta_\omega\eta, \quad \mathcal{K}_1(0) = 1, \quad \text{and} \quad \mathcal{K}_1'(0)\eta = -\eta.$$

Hence we have

$$\mathcal{B}(0)\eta = -\gamma\mathcal{D}(0)\mathcal{T}(0)\mathcal{L}(0)\eta = \frac{\gamma}{n-1}\frac{\partial}{\partial r}\Pi(\Delta_\omega\eta)\bigg|_{r=1}. \tag{5.7}$$

We denote $u^s_{\varepsilon,\eta} = \Theta^*_{\varepsilon\eta}\sigma_s - \bar{\sigma}$. Then we have

$$\mathcal{A}(\varepsilon\eta)u^s_{\varepsilon,\eta} = \mathcal{F}\big(u^s_{\varepsilon,\eta} + \bar{\sigma}\big),$$

so that

$$\big[\mathcal{A}(\varepsilon\eta) - \mathcal{A}(0)\big]u^s_{\varepsilon,\eta} + \mathcal{A}(0)\big(u^s_{\varepsilon,\eta} - u_s\big) = \mathcal{F}\big(u^s_{\varepsilon,\eta} + \bar{\sigma}\big) - \mathcal{F}(u_s + \bar{\sigma}).$$

Dividing both sides with ε and letting $\varepsilon \to 0$, we get

$$\big[\mathcal{A}'(0)\eta\big]u_s + \mathcal{A}(0)\big[\mathcal{M}(0,\sigma_s)\Pi(\eta)\big] = \mathcal{F}'(\sigma_s)\big[\mathcal{M}(0,\sigma_s)\Pi(\eta)\big].$$

Here we used the fact that $\lim_{\varepsilon\to 0}\varepsilon^{-1}(u^s_{\varepsilon,\eta} - u_s) = \mathcal{M}(0,\sigma_s)\Pi(\eta)$ $(=\phi(r-1)\sigma_s'(r)\Pi(\eta))$. Hence,

$$\begin{aligned}
\big[\mathcal{A}'(0)\eta\big]u_s &= -\mathcal{A}(0)\big[\mathcal{M}(0,\sigma_s)\Pi(\eta)\big] + \mathcal{F}'(\sigma_s)\big[\mathcal{M}(0,\sigma_s)\Pi(\eta)\big] \\
&= -\Delta\big[\phi(r-1)\sigma_s'(r)\Pi(\eta)\big] + f'\big(\sigma_s(r)\big)\phi(r-1)\sigma_s'(r)\Pi(\eta) \\
&= -\big[\Delta - f'\big(\sigma_s(r)\big)\big]\big[\phi(r-1)\sigma_s'(r)\Pi(\eta)\big].
\end{aligned} \tag{5.8}$$

To compute $\mathbb{F}_0'(U_s)$, we first note that since $\mathcal{P}(\rho,u)v = \mathcal{M}(\rho,u)\Pi(\mathcal{D}(\rho)v)$, we have

$$\begin{aligned}
\mathcal{F}_2(\rho,u) &= -c^{-1}\mathcal{F}(u+\bar{\sigma}) - \gamma\mathcal{P}(\rho,u+\bar{\sigma})\mathcal{T}(\rho)\mathcal{K}_1(\rho) - \mathcal{P}(\rho,u+\bar{\sigma})\mathcal{S}(\rho)\mathcal{G}(u+\bar{\sigma}) \\
&= -c^{-1}\mathcal{F}(u+\bar{\sigma}) - \gamma\mathcal{M}(\rho,u+\bar{\sigma})\Pi\big[\mathcal{D}(\rho)\mathcal{T}(\rho)\mathcal{K}_1(\rho)\big] \\
&\quad - \mathcal{M}(\rho,u+\bar{\sigma})\Pi\big[\mathcal{D}(\rho)\mathcal{S}(\rho)\mathcal{G}(u+\bar{\sigma})\big].
\end{aligned}$$

Differentiating this expression in u at $(\rho,u) = (0,u_s)$ yields

$$\begin{aligned}
D_u\mathcal{F}_2(0,u_s)v &= -c^{-1}f'\big(\sigma_s(r)\big)v - \gamma\mathcal{M}(0,v)\Pi\big[\mathcal{D}(0)\mathcal{T}(0)\mathcal{K}_1(0)\big] \\
&\quad - \mathcal{M}(0,v)\Pi\big[\mathcal{D}(0)\mathcal{S}(0)g\big(\sigma_s(r)\big)\big] - \mathcal{M}(0,u_s)\Pi\big[\mathcal{D}(0)\mathcal{S}(0)g'\big(\sigma_s(r)\big)v\big].
\end{aligned}$$

We have $\mathcal{D}(0)\mathcal{T}(0)\mathcal{K}_1(0) = \mathcal{D}(0)\mathcal{T}(0)1 = \mathcal{D}(0)1 = 0$, and, by denoting $v_s(r) = p_s(r) - p_s(1)$, $\mathcal{D}(0)\mathcal{S}(0)g(\sigma_s(r)) = \mathcal{D}(0)v_s = p_s'(1) = 0$. Hence,

$$\begin{aligned}
D_u\mathcal{F}_2(0,u_s)v &= -c^{-1}f'\big(\sigma_s(r)\big)v - \mathcal{M}(0,u_s)\Pi\big[\mathcal{D}(0)\mathcal{S}(0)g'\big(\sigma_s(r)\big)v\big] \\
&= -c^{-1}f'\big(\sigma_s(r)\big)v - \phi(r-1)\sigma_s'(r)\Pi\big[\mathcal{D}(0)\mathcal{S}(0)g'\big(\sigma_s(r)\big)v\big].
\end{aligned} \tag{5.9}$$

In order to compute $D_\rho\mathcal{F}_2(0,u_s)$ we write

$$\begin{aligned}
\mathcal{D}(0)\mathcal{T}(0)\mathcal{K}_1'(0)\eta = -\mathcal{D}(0)\mathcal{T}(0)\eta = -\mathcal{D}(0)\Pi(\eta) = -\frac{\partial\Pi(\eta)}{\partial r}\bigg|_{r=1}, \\
\mathcal{D}(0)\big[\mathcal{T}'(0)\eta\big]\mathcal{K}_1(0) = \mathcal{D}(0)\big[\mathcal{T}'(0)\eta\big]1 = 0 \quad \big(\text{because } \mathcal{T}(\varepsilon\eta)1 = \mathcal{T}(0)1 = 1\big), \\
\big[\mathcal{D}'(0)\eta\big]\mathcal{T}(0)\mathcal{K}_1(0) = \big[\mathcal{D}'(0)\eta\big]\mathcal{T}(0)1 = \big[\mathcal{D}'(0)\eta\big]1 = 0 \quad \big(\text{because } \mathcal{D}(\varepsilon\eta)1 = \mathcal{D}(0)1 = 0\big), \\
\big[\mathcal{D}'(0)\eta\big]\mathcal{S}(0)g\big(\sigma_s(r)\big) = \big[\mathcal{D}'(0)\eta\big]v_s = p_s'(1)\eta = 0, \\
\mathcal{D}(0)\big[\mathcal{S}'(0)\eta\big]g\big(\sigma_s(r)\big) = \mathcal{D}(0)\mathcal{S}(0)\big[\mathcal{A}'(0)\eta\big]\mathcal{S}(0)g\big(\sigma_s(r)\big) = \mathcal{D}(0)\mathcal{S}(0)\big[\mathcal{A}'(0)\eta\big]v_s.
\end{aligned} \tag{5.10}$$

In getting (5.10) we used the identity $\mathcal{S}'(0)\eta = \mathcal{S}(0)[\mathcal{A}'(0)\eta]\mathcal{S}(0)$, which follows from the fact that $\mathcal{A}(\rho)\mathcal{S}(\rho) = -\mathrm{id}$ for any $\rho \in C^2(\mathbb{S}^{n-1})$. By a similar argument as in the proof of (5.8) we see that

$$\begin{aligned}\big[\mathcal{A}'(0)\eta\big]v_s &= -\mathcal{A}(0)\big[\mathcal{M}(0,p_s)\Pi(\eta)\big] - \mathcal{G}'(\sigma_s)\big[\mathcal{M}(0,\sigma_s)\Pi(\eta)\big]\\ &= -\Delta\big[\phi(r-1)p_s'(r)\Pi(\eta)\big] - g'\big(\sigma_s(r)\big)\phi(r-1)\sigma_s'(r)\Pi(\eta).\end{aligned} \tag{5.11}$$

Substituting (5.11) into (5.10) we get

$$\begin{aligned}\mathcal{D}(0)\big[\mathcal{S}'(0)\eta\big]g\big(\sigma_s(r)\big) &= \mathcal{D}(0)\big[\phi(r-1)p_s'(r)\Pi(\eta)\big] - \mathcal{D}(0)\mathcal{S}(0)\big[g'\big(\sigma_s(r)\big)\phi(r-1)\sigma_s'(r)\Pi(\eta)\big]\\ &= -g(\bar\sigma)\eta - \mathcal{D}(0)\mathcal{S}(0)\big[g'\big(\sigma_s(r)\big)\phi(r-1)\sigma_s'(r)\Pi(\eta)\big].\end{aligned}$$

Using these results and the relations $\mathcal{D}(0)\mathcal{T}(0)\mathcal{K}_1(0) = 0$ and $\mathcal{D}(0)\mathcal{S}(0)g(\sigma_s(r)) = 0$, we see that

$$\begin{aligned}D_\rho\mathcal{F}_2(0,u_s)\eta &= -\gamma D_\rho\mathcal{M}(0,\sigma_s)\eta\cdot\Pi\big(\mathcal{D}(0)\mathcal{T}(0)\mathcal{K}_1(0)\big) - \gamma\mathcal{M}(0,\sigma_s)\Pi\big(\mathcal{D}(0)\mathcal{T}(0)\mathcal{K}_1'(0)\eta\big)\\ &\quad - \gamma\mathcal{M}(0,\sigma_s)\Pi\big(\mathcal{D}(0)\big[\mathcal{T}'(0)\eta\big]\mathcal{K}_1(0)\big) - \gamma\mathcal{M}(0,\sigma_s)\Pi\big(\big[\mathcal{D}'(0)\eta\big]\mathcal{T}(0)\mathcal{K}_1(0)\big)\\ &\quad - D_\rho\mathcal{M}(0,\sigma_s)\eta\cdot\Pi\big(\mathcal{D}(0)\mathcal{S}(0)g\big(\sigma_s(r)\big)\big) - \mathcal{M}(0,\sigma_s)\Pi\big(\mathcal{D}(0)\big[\mathcal{S}'(0)\eta\big]g\big(\sigma_s(r)\big)\big)\\ &\quad - \mathcal{M}(0,\sigma_s)\Pi\big(\big[\mathcal{D}'(0)\eta\big]\mathcal{S}(0)g\big(\sigma_s(r)\big)\big)\\ &= \gamma\phi(r-1)\sigma_s'(r)\Pi\left(\frac{\partial\Pi(\eta)}{\partial r}\bigg|_{r=1}\right) + \phi(r-1)\sigma_s'(r)\Pi\big(\mathcal{D}(0)\mathcal{S}(0)\big[g'\big(\sigma_s(r)\big)\\ &\quad \times\phi(r-1)\sigma_s'(r)\Pi(\eta)\big]\big) + g(\bar\sigma)\phi(r-1)\sigma_s'(r)\Pi\big(\eta\big).\end{aligned} \tag{5.12}$$

Finally, differentiating $\mathcal{G}_2(\rho,u) = -\gamma\mathcal{D}(\rho)\mathcal{T}(\rho)\mathcal{K}_1(\rho) - \mathcal{D}(\rho)\mathcal{S}(\rho)\mathcal{G}(u+\bar\sigma)$ in u at $(\rho,u) = (0,u_s)$ yields

$$D_u\mathcal{G}_2(0,u_s)v = -\mathcal{D}(0)\mathcal{S}(0)\big[g'\big(\sigma_s(r)\big)v\big], \tag{5.13}$$

and differentiating in ρ gives

$$\begin{aligned}D_\rho\mathcal{G}_2(0,u_s)\eta &= -\gamma\mathcal{D}(0)\mathcal{T}(0)\mathcal{K}_1'(0)\eta - \gamma\mathcal{D}(0)\big[\mathcal{T}'(0)\eta\big]\mathcal{K}_1(0) - \gamma\big[\mathcal{D}'(0)\eta\big]\mathcal{T}(0)\mathcal{K}_1(0)\\ &\quad - \big[\mathcal{D}'(0)\eta\big]\mathcal{S}(0)g\big(\sigma_s(r)\big) - \mathcal{D}(0)\big[\mathcal{S}'(0)\eta\big]g\big(\sigma_s(r)\big)\\ &= \gamma\frac{\partial\Pi(\eta)}{\partial r}\bigg|_{r=1} + \mathcal{D}(0)\mathcal{S}(0)\big[g'\big(\sigma_s(r)\big)\phi(r-1)\sigma_s'(r)\Pi(\eta)\big] + g(\bar\sigma)\eta.\end{aligned} \tag{5.14}$$

From (5.1)–(5.5), (5.7)–(5.9) and (5.12)–(5.14) we obtain

$$\mathbb{F}'(U_s) = \begin{pmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{pmatrix}, \tag{5.15}$$

where, by denoting $m(r) = \phi(r-1)\sigma_s'(r)$,

$$\begin{aligned}\mathcal{A}_{11}v &= c^{-1}\big[\Delta - f'\big(\sigma_s(r)\big)\big]v - m(r)\Pi\big[\mathcal{D}(0)\mathcal{S}(0)\big[g'\big(\sigma_s(r)\big)v\big]\big],\\ \mathcal{A}_{12}\eta &= m(r)\Pi\left[\gamma\frac{\partial}{\partial r}\Pi\left(\eta + \frac{1}{n-1}\Delta_\omega\eta\right)\bigg|_{r=1} + g(\bar\sigma)\eta\right] - c^{-1}\big[\Delta - f'\big(\sigma_s(r)\big)\big]\big[m(r)\Pi(\eta)\big]\\ &\quad + m(r)\Pi\big[\mathcal{D}(0)\mathcal{S}(0)\big[g'\big(\sigma_s(r)\big)m(r)\Pi(\eta)\big]\big],\end{aligned}$$

$$\mathcal{A}_{21}v = -\mathcal{D}(0)\mathcal{S}(0)\big[g'\big(\sigma_s(r)\big)v\big],$$
$$\mathcal{A}_{22}\eta = \gamma\frac{\partial}{\partial r}\Pi\left(\eta + \frac{1}{n-1}\Delta_\omega\eta\right)\bigg|_{r=1} + g(\bar\sigma)\eta + \mathcal{D}(0)\mathcal{S}(0)\big[g'\big(\sigma_s(r)\big)m(r)\Pi(\eta)\big].$$

We summarize the above result in the following lemma:

Lemma 5.1. *The Fréchet derivative $\mathbb{F}'(U_s)$ is given by* (5.15)*.*

In Section 4 we proved, by using the relation (4.16), that W_j $(j = 1, 2, \dots, n)$ given in (4.18) are eigenvectors of $\mathbb{F}'(U_s)$ corresponding to the eigenvalue 0. We can easily reprove this result by using the expression (5.15) of $\mathbb{F}'(U_s)$.

Lemma 5.2. *The operator $\mathbb{F}'(U_s)$, regarded as an unbounded linear operator in X with domain X_0, is a sectorial operator.*

Proof. Recalling (5.1), we see that $\mathbb{F}'(U_s) = A + B$, where $A = \mathbb{A}(U_s)$ and $BV = [\mathbb{A}'(U_s)V]U_s + \mathbb{F}_0'(U_s)V$ for $V \in X_0$. By Lemma 4.1 of [14] we know that A is a sectorial operator in X with domain X_0. Next we consider B. Since $\mathbb{F}_0 \in C^\infty(\mathcal{O}, Y)$, we have $\mathbb{F}_0'(U_s) \in L(X_0, Y)$. Besides, from the second relation in (5.2) and the result obtained in (5.8) we easily see that the mapping $V \to [\mathbb{A}'(U_s)V]U_s$ also belong to $L(X_0, Y)$. Hence, in conclusion we have $B \in L(X_0, Y)$. Since Y is clearly an intermediate space between X_0 and X, by a standard result we get the desired assertion. □

By a standard perturbation result, we have

Corollary 5.3. *If the neighborhood $\mathcal{O}$ of U_s (in X_0) is sufficiently small, then for any $U \in \mathcal{O}$, $\mathbb{F}'(U)$ is a sectorial operator.*

Later on we shall assume that the number δ is so small that the open set $\mathcal{O}$ defined in the end of Section 3 satisfies the condition of the above corollary.

6. The spectrum of $\mathbb{F}'(U_s)$

Given a closed linear operator B in a Banach space X, we denote by $\rho(B)$ and $\sigma(B)$ the resolvent set and the spectrum of B, respectively. In the sequel we study $\sigma(\mathbb{F}'(U_s))$.

We introduce the operator $\mathcal{A}_0 : W^{m-1,q}(\mathbb{B}^n) \to W^{m-3,q}(\mathbb{B}^n)$ by

$$\mathcal{A}_0 v = \big[\Delta - f'\big(\sigma_s(r)\big)\big]v \quad \text{for } v \in W^{m-1,q}\big(\mathbb{B}^n\big),$$

the operator $Q : B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \to W^{m,q}(\mathbb{B}^n)$ by

$$Q\eta = m(r)\Pi(\eta) = \phi(r-1)\sigma_s'(r)\Pi(\eta) \quad \text{for } \eta \in B_{qq}^{m-1/q}\big(\mathbb{S}^{n-1}\big),$$

and the operator $\mathcal{J} : W^{m-1,q}(\mathbb{B}^n) \to B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ by

$$\mathcal{J}v = -\mathcal{D}(0)\mathcal{S}(0)\big[g'\big(\sigma_s(r)\big)v\big] = \frac{\partial}{\partial r}\big\{\Delta^{-1}\big[g'\big(\sigma_s(r)\big)v\big]\big\}\Big|_{r=1} \quad \text{for } v \in W^{m-1,q}\big(\mathbb{B}^n\big).$$

Here Δ^{-1} denotes the inverse of the Laplacian under the homogeneous Dirichlet boundary condition. Let $\Pi_0 : B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \to W^{m,q}(\mathbb{B}^n)$ be the operator $\Pi_0(\eta) = v$, where for given $\eta \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$, $v \in W^{m,q}(\mathbb{B}^n)$ is the solution of the boundary value problem

$$\Delta v - f'\big(\sigma_s(r)\big)v = 0 \quad \text{in } \mathbb{B}^n, \quad \text{and} \quad v = \eta \quad \text{on } \mathbb{S}^{n-1}.$$

Note that this definition implies that $\mathcal{A}_0\Pi_0 = 0$. We define $\mathcal{B}_\gamma : B_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \to B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$ by

$$\begin{aligned}\mathcal{B}_\gamma\eta &= \gamma\frac{\partial}{\partial r}\left\{\Pi\left(\eta+\frac{1}{n-1}\Delta_\omega\eta\right)\right\}\Big|_{r=1} + g(\bar\sigma)\eta - \sigma_s'(1)\mathcal{J}\Pi_0(\eta)\\ &= \frac{\partial}{\partial r}\left\{\gamma\Pi\left(\eta+\frac{1}{n-1}\Delta_\omega\eta\right) - \sigma_s'(1)\Delta^{-1}\big(g'\big(\sigma_s(r)\big)\Pi_0(\eta)\big)\right\}\Big|_{r=1} + g(\bar\sigma)\eta \quad \text{for } \eta\in B_{qq}^{m-1/q}\big(\mathbb{S}^{n-1}\big).\end{aligned}$$

Finally we define the operators $\mathbb{M}: X_0 \to X$ and $\mathbb{T}: X \to X$ respectively by

$$\mathbb{M} = \begin{pmatrix} c^{-1}\mathcal{A}_0 + \sigma_s'(1)\Pi_0\mathcal{J} & \sigma_s'(1)\Pi_0\mathcal{B}_\gamma \\ \mathcal{J} & \mathcal{B}_\gamma\end{pmatrix} \quad\text{and}\quad \mathbb{T} = \begin{pmatrix} I & \sigma_s'(1)\Pi_0 - Q \\ 0 & I\end{pmatrix}.$$

Here the first I in $\mathbb{T}$ represents the identity operator in $W^{m-3,q}(\mathbb{B}^n)$, while the second I in $\mathbb{T}$ represents the identity operator in $B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$. Note that $(\sigma_s'(1)\Pi_0 - Q)\eta|_{\mathbb{S}^{n-1}} = 0$ for any $\eta \in B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$, so that $\mathbb{T}$ maps X_0 to X_0.

Lemma 6.1. *For $V \in X_0$ and $\lambda \in \mathbb{C}$, the relation $\mathbb{F}'(U_s)V = \lambda V$ holds if and only if the relations $\mathbb{M}W = \lambda W$ and $W = \mathbb{T}V$ hold.*

Proof. Clearly,

$$\mathcal{A}_{11}v = c^{-1}\mathcal{A}_0 v + Q\mathcal{J}v, \qquad \mathcal{A}_{12}\eta = -c^{-1}\mathcal{A}_0 Q\eta + Q\left(\mathcal{B}_\gamma + \sigma_s'(1)\mathcal{J}\Pi_0 - \mathcal{J}Q\right)\eta,$$
$$\mathcal{A}_{21}v = \mathcal{J}v, \qquad \mathcal{A}_{22}\eta = \big(\mathcal{B}_\gamma + \sigma_s'(1)\mathcal{J}\Pi_0 - \mathcal{J}Q\big)\eta.$$

Using these relations and the fact that $\mathcal{A}_0\Pi_0 = 0$ we can easily verify that

$$\begin{pmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{pmatrix} = \begin{pmatrix} I & Q-\sigma_s'(1)\Pi_0\\ 0 & I\end{pmatrix}\begin{pmatrix} c^{-1}\mathcal{A}_0 + \sigma_s'(1)\Pi_0\mathcal{J} & \sigma_s'(1)\Pi_0\mathcal{B}_\gamma \\ \mathcal{J} & \mathcal{B}_\gamma\end{pmatrix}\begin{pmatrix} I & \sigma_s'(1)\Pi_0 - Q \\ 0 & I\end{pmatrix},$$

or $\mathbb{F}'(U_s) = \mathbb{T}^{-1}\mathbb{M}\mathbb{T}$. From this relation the desired assertion follows immediately. $\square$

Since X_0 is clearly compactly embedded in X, by Lemma 5.2 we see that $\sigma(\mathbb{F}'(U_s))$ consists entirely of eigenvalues. Hence, by Lemma 6.1 we have

Corollary 6.2. $\sigma(\mathbb{F}'(U_s)) = \sigma(\mathbb{M})$.

We shall see that for sufficiently small c, $\sigma(\mathcal{B}_\gamma)$ plays a major role in determining $\sigma(\mathbb{M})$. Hence, in the sequel we first compute $\sigma(\mathcal{B}_\gamma)$. To this end we introduce some notation and recall some results of [15]. For every nonnegative integer k, let $Y_{kl}(\omega)$, $l = 1, 2, \dots, d_k$, be the normalized orthogonal basis of the space of all spherical harmonics of degree k, where d_k is the dimension of this space, i.e.

$$d_0 = 1, \qquad d_1 = n, \qquad d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \quad (k \geqslant 2).$$

It is well known that

$$\Delta_\omega Y_{kl}(\omega) = -\lambda_k Y_{kl}(\omega), \qquad \lambda_k = k^2 + (n-2)k \quad (k = 0, 1, 2, \dots),$$

and λ_k ($k = 0, 1, 2, \dots$) are the all eigenvalues of Δ_ω. We denote

$$a_k = 2k + n - 1 \geqslant n - 1,$$

and denote by $\bar{u}_k(r)$ the solution of the initial value problem

$$\begin{cases} \bar{u}_k''(r) + \dfrac{a_k}{r}\bar{u}_k'(r) = f'\big(\sigma_s(r)\big)\bar{u}_k(r), \\ \bar{u}_k(0) = 1, \quad \bar{u}_k'(0) = 0. \end{cases}$$

By using some ODE techniques we can show that this problem has a unique solution for all $r \in [0, R^*)$, where $[0, R^*)$ is the maximal existence interval of $\sigma_s(r)$. We also denote

$$\gamma_1 = 0, \qquad \gamma_k = \frac{n-1}{(\lambda_k - n + 1)k}\left[g(\bar{\sigma}) - \frac{\sigma_0'(1)}{\bar{u}_k(1)}\int_0^1 g'\big(\sigma_0(\rho)\big)\bar{u}_k(\rho)\rho^{a_k}\,d\rho\right] \quad (k \geqslant 2).$$

From [15] we know that $\gamma_1, \gamma_2, \dots$ are the all eigenvalues of the linearization of the stationary version of the system (1.1)–(1.5) at the radially symmetric stationary solution $(\sigma_s, p_s, \Omega_s)$, $\gamma_k > 0$ for $k \geqslant 2$, and $\lim_{k\to\infty}\gamma_k = 0$. Next we denote

$$\alpha_{k,\gamma} = -\frac{(\lambda_k - n + 1)k}{n-1}(\gamma - \gamma_k), \quad k = 1, 2, \dots.$$

Note that $\alpha_{1,\gamma} = 0$ (because $\lambda_1 = n-1$) and $\alpha_{k,\gamma} \sim -\gamma k^3/(n-1)$ as $k \to \infty$. Finally, we denote

$$\alpha_{0,\gamma} = \alpha_0 = g(\bar{\sigma}) - \frac{\sigma_0'(R_s)}{\bar{u}_0(R_s)R_s^{n-1}}\int_0^{R_s} g'\big(\sigma_0(r)\big)\bar{u}_0(r)r^{n-1}\,dr.$$

From [15] we know that $\alpha_0 < 0$.

Lemma 6.3. *$\mathcal{B}_\gamma$ is a Fourier multiplication operator of the following form: For any $\eta(\omega) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} b_{kl}Y_{kl}(\omega) \in C^\infty(\mathbb{S}^{n-1})$,*

$$\mathcal{B}_\gamma\eta(\omega) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\alpha_{k,\gamma}b_{kl}Y_{kl}(\omega). \tag{6.1}$$

As a result, we have $\sigma(\mathcal{B}_\gamma) = \{\alpha_{k,\gamma}\colon k \in \mathbb{N},\ k \geqslant 2\} \cup \{0, \alpha_0\}$.

Proof. It can be easily seen that $\mathcal{B}_\gamma$ has the same expression as that introduced in [16] with the same notation (but notice that $\mathcal{B}_\gamma$ in [16] is a mapping from $C^{m+\mu}(\mathbb{S}^{n-1})$ to $C^{m-3+\mu}(\mathbb{S}^{n-1})$ for some inter m and $0 < \mu < 1$). Hence, by a similar calculation as in [16] we get (6.1). □

Corollary 6.4. $\dim\operatorname{Ker}\mathbb{F}'(U_s) = n$.

Proof. By Lemma 6.1, $\mathbb{F}'(U_s)V = 0$ if and only if $\mathbb{M}W = 0$, where $W = \mathbb{T}V$. Let $W = \binom{u}{\eta}$. Then it is obvious that $\mathbb{M}W = 0$ if and only if $\mathcal{A}_0 u = 0$ and $\mathcal{J}u + \mathcal{B}_\gamma\eta = 0$. Since $\mathcal{A}_0 u = 0$ implies that $u = 0$, we see that $\mathbb{M}W = 0$ if and only if $W = \binom{0}{\eta}$ and $\mathcal{B}_\gamma\eta = 0$. Hence, from (6.1) we immediately get the desired assertion. □

Lemma 6.5. *For any $\gamma > 0$ there exists corresponding $c_0 > 0$ such that for any $0 < c \leqslant c_0$, $\mathbb{M}$ has eigenvalues $\lambda_{k,\gamma} = \alpha_{k,\gamma} + c\mu_{k,\gamma}(c)$ $(k = 0, 1, 2, \dots)$, where $\mu_{k,\gamma}(c)$ are bounded continuous functions in $0 < c \leqslant c_0$. Moreover, for each k the corresponding eigenvectors of $\mathbb{M}$ have the form $\binom{c a_{k,\gamma}(r,c)}{1}Y_{kl}(\omega)$ $(l = 1, 2, \dots, d_k)$,* *where $a_{k,\gamma}(r,c)$ are smooth functions in $r \in [0,1]$ and are bounded and continuous in $0 < c \leqslant c_0$. In particular, $\mu_{1,\gamma}(c) = 0$ and $\lambda_{1,\gamma} = \alpha_{1,\gamma} = 0$.*

Proof. Let $U = \binom{ca_{k,\gamma}(r,c)}{1} Y_{kl}(\omega)$. Then the relations $\mathbb{M}U = \lambda_{k,\gamma} U$ and $\lambda_{k,\gamma} = \alpha_{k,\gamma} + c\mu_{k,\gamma}(c)$ hold if and only if the following relations hold:

$$\mathcal{A}_0(a_{k,\gamma} Y_{kl}) + c\sigma_s'(1)\Pi_0 \mathcal{J}(a_{k,\gamma} Y_{kl}) + \sigma_s'(1)\Pi_0 \mathcal{B}_\gamma(Y_{kl}) = c\alpha_{k,\gamma} a_{k,\gamma} Y_{kl} + c^2 \mu_{k,\gamma} a_{k,\gamma} Y_{kl}, \tag{6.2}$$

$$c\mathcal{J}(a_{k,\gamma} Y_{kl}) + \mathcal{B}_\gamma(Y_{kl}) = \alpha_{k,\gamma} Y_{kl} + c\mu_{k,\gamma} Y_{kl}. \tag{6.3}$$

Let $\mathcal{L}_k$ be the second-order differential operator $\mathcal{L}_k u(r) = u''(r) + \frac{n-1}{r} u'(r) - \frac{\lambda_k}{r^2} u(r)$, and $\mathcal{J}_k$ be the operator $u \to v_k'(1)$, where for a given continuous function $u = u(r)$ $(0 \leqslant r \leqslant 1)$, $v = v_k(r)$ is the solution of the boundary value problem:

$$\begin{cases} v''(r) + \dfrac{n-1}{r} v'(r) - \dfrac{\lambda_k}{r^2} v(r) = g'\big(\sigma_s(r)\big) u(r), & 0 < r < 1, \\ v'(0) = 0, \quad v(1) = 0. \end{cases}$$

Then we have $\mathcal{A}_0(a_{k,\gamma} Y_{kl}) = \mathcal{L}_k(a_{k,\gamma}) Y_{kl}$ and $\mathcal{J}(a_{k,\gamma} Y_{kl}) = \mathcal{J}_k(a_{k,\gamma}) Y_{kl}$. Besides, it can be easily seen that $\Pi_0(Y_{kl}) = w_k(r) Y_{kl}$, where $w_k(r)$ $(0 \leqslant r \leqslant 1)$ is the solution of the boundary value problem:

$$\begin{cases} w_k''(r) + \dfrac{n-1}{r} w_k'(r) - \left(\dfrac{\lambda_k}{r^2} + f'\big(\sigma_s(r)\big)\right) w_k(r) = 0, & 0 < r < 1, \\ w_k'(0) = 0, \quad w_k(1) = 0. \end{cases}$$

Using these facts and the relation $\mathcal{B}_\gamma(Y_{kl}) = \alpha_{k,\gamma} Y_{kl}$ (cf. (6.1)) we see that (6.2) and (6.3) reduce to the following system of equations:

$$\mathcal{L}_k(a_{k,\gamma}) + c\sigma_s'(1)\mathcal{J}_k(a_{k,\gamma}) w_k(r) + \sigma_s'(1)\alpha_{k,\gamma} w_k(r) = c\alpha_{k,\gamma} a_{k,\gamma} + c^2 \mu_{k,\gamma} a_{k,\gamma},$$

$$\mu_{k,\gamma} = \mathcal{J}_k(a_{k,\gamma}),$$

which can be further reduced to the following scalar equation in $a_{k,\gamma}$:

$$\mathcal{L}_k(a_{k,\gamma}) = -c\sigma_s'(1)\mathcal{J}_k(a_{k,\gamma}) w_k(r) + c\alpha_{k,\gamma} a_{k,\gamma} + c^2 a_{k,\gamma} \mathcal{J}_k(a_{k,\gamma}) - \sigma_s'(1)\alpha_{k,\gamma} w_k(r).$$

By using a standard fixed point argument we can easily show that for c sufficiently small this equation complemented with the boundary value conditions $\frac{\partial a_{k,\gamma}}{\partial r}|_{r=0} = 0$ and $a_{k,\gamma}|_{r=1} = 0$ has a unique solution. Hence the system of Eqs. (6.2) and (6.3) has a solution which is unique up to a constant factor. Finally, using Lemma 6.1 and (4.18) we can easily deduce that $\mu_{1,\gamma}(c) = 0$ and $\lambda_{1,\gamma} = \alpha_{1,\gamma} = 0$. This completes the proof. $\square$

We denote

$$\gamma_* = \max_{k \geqslant 2} \gamma_k \quad \text{and} \quad \alpha_\gamma^* = \max\Big\{\alpha_0, \max_{k \geqslant 2} \alpha_{k,\gamma}\Big\}.$$

Since $\gamma_k > 0$, $\lim_{k\to\infty} \gamma_k = 0$ and $\lim_{k\to\infty} \alpha_{k,\gamma} = -\infty$, γ_* and α_γ^* are both well defined. Clearly, we have $\alpha_\gamma^* < 0$ for $\gamma > \gamma_*$, and $\alpha_\gamma^* > 0$ for $0 < \gamma < \gamma_*$.

Lemma 6.6. *Given $\gamma > \gamma_*$, there exists corresponding $c_0 > 0$ such that for any $0 < c \leqslant c_0$ and any $\lambda \in \mathbb{C} \setminus \{0\}$ satisfying* $\operatorname{Re}\lambda \geqslant \frac{1}{2}\alpha_\gamma^*$, *there holds $\lambda \in \rho(\mathbb{M})$, or equivalently,*

$$\sup\bigl\{\operatorname{Re}\lambda\colon\ \lambda \in \sigma(\mathbb{M}) \setminus \{0\}\bigr\} \leqslant \frac{1}{2}\alpha_\gamma^* < 0.$$

Proof. Let $\gamma > \gamma_*$ be given. We denote

$$\mathbb{M}_0 = \begin{pmatrix} c^{-1}\mathcal{A}_0 & 0 \\ \mathcal{J} & \mathcal{B}_\gamma \end{pmatrix}, \qquad \mathbb{N} = \begin{pmatrix} \sigma_s'(1)\Pi_0\mathcal{J} & \sigma_s'(1)\Pi_0\mathcal{B}_\gamma \\ 0 & 0 \end{pmatrix}.$$

Then $\mathbb{M}_0 \in L(X_0, X)$, $\mathbb{N} \in L(X_0, X)$, and $\mathbb{M} = \mathbb{M}_0 + \mathbb{N}$. Since $f'(\sigma_s(r)) \geqslant 0$, From the standard theory of elliptic partial differential equations of the second order we know that all eigenvalues of $\mathcal{A}_0$ (regarded as an unbounded closed linear operator in $W^{m-3,q}(\mathbb{B}^n)$ with domain $W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)$) are negative and they make up a decreasing sequence tending to $-\infty$. Let ν_1 be the largest eigenvalue of $\mathcal{A}_0$, and let $c_0 = \nu_1/\alpha_\gamma^*$. Since $\nu_1 < 0$ and $\alpha^* < 0$, we have $c_0 > 0$. For any $0 < c \leqslant c_0$ and any $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Re}\lambda \geqslant \frac{1}{2}\alpha_\gamma^*$ we have $\operatorname{Re}(c\lambda) \geqslant \frac{1}{2}\nu_1$, so that both $\lambda I - c^{-1}\mathcal{A}_0 = c^{-1}(c\lambda I - \mathcal{A}_0)$ and $\lambda I - \mathcal{B}_\gamma$ are invertible, which implies that $\lambda I - \mathbb{M}_0$ is invertible. In fact,

$$(\lambda I - \mathbb{M}_0)^{-1} = \begin{pmatrix} (\lambda I - c^{-1}\mathcal{A}_0)^{-1} & 0 \\ (\lambda I - \mathcal{B}_\gamma)^{-1}\mathcal{J}(\lambda I - c^{-1}\mathcal{A}_0)^{-1} & (\lambda I - \mathcal{B}_\gamma)^{-1} \end{pmatrix}.$$

Hence

$$\lambda I - \mathbb{M} = (\lambda I - \mathbb{M}_0) - \mathbb{N} = (\lambda I - \mathbb{M}_0)(I - c\mathbb{K}),$$

where

$$\begin{aligned} \mathbb{K} &= c^{-1}(\lambda I - \mathbb{M}_0)^{-1}\mathbb{N} \\ &= \begin{pmatrix} (c\lambda I - \mathcal{A}_0)^{-1}\sigma_s'(1)\Pi_0\mathcal{J} & (c\lambda I - \mathcal{A}_0)^{-1}\sigma_s'(1)\Pi_0\mathcal{B}_\gamma \\ (\lambda I - \mathcal{B}_\gamma)^{-1}\mathcal{J}(c\lambda I - \mathcal{A}_0)^{-1}\sigma_s'(1)\Pi_0\mathcal{J} & (\lambda I - \mathcal{B}_\gamma)^{-1}\mathcal{J}(c\lambda I - \mathcal{A}_0)^{-1}\sigma_s'(1)\Pi_0\mathcal{B}_\gamma \end{pmatrix}. \end{aligned}$$

Since $\mathcal{A}_0$ is a self-adjoint sectorial operator and ν_1 is the maximal eigenvalue of $\mathcal{A}_0$, we have

$$\bigl\|(c\lambda I - \mathcal{A}_0)^{-1}\bigr\|_{L(W^{m-3,q}(\mathbb{B}^n), W^{m-3,q}(\mathbb{B}^n))} \leqslant \frac{C}{|c\lambda - \nu_1|} \leqslant 2C/\nu_1,$$

where C is a constant independent of c and λ. Using this fact, the identity

$$\mathcal{A}_0(c\lambda I - \mathcal{A}_0)^{-1} = c\lambda(c\lambda I - \mathcal{A}_0)^{-1} - I,$$

and the Agmon–Douglis–Nirenberg inequality, we obtain

$$\begin{aligned} &\bigl\|(c\lambda I - \mathcal{A}_0)^{-1}\bigr\|_{L(W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n), W^{m-3,q}(\mathbb{B}^n))} \\ &\quad \leqslant C\bigl[\bigl\|(c\lambda I - \mathcal{A}_0)^{-1}\bigr\|_{L(W^{m-3,q}(\mathbb{B}^n), W^{m-3,q}(\mathbb{B}^n))} + \bigl\|\mathcal{A}_0(c\lambda I - \mathcal{A}_0)^{-1}\bigr\|_{L(W^{m-3,q}(\mathbb{B}^n), W^{m-3,q}(\mathbb{B}^n))}\bigr] \\ &\quad \leqslant C + \frac{C|c\lambda|}{|c\lambda - \nu_1|} \leqslant C. \end{aligned}$$

Since $\mathcal{B}_\gamma$ is a third-order pseudo-differential operator on $\mathbb{S}^{n-1}$ of the elliptic type (cf. [16,21]), by a similar argument we have

$$\left\|(\lambda I-\mathcal{B}_\gamma)^{-1}\right\|_{L(B_{qq}^{m-1/q}(\mathbb{S}^{n-1}),\,B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1}))}\leqslant C.$$

Using these estimates we can easily show that

$$\|\mathbb{K}\|_{L(X_0,X_0)}\leqslant C$$

for any $0<c\leqslant c_0$ and any $\lambda\in\mathbb{C}$ such that $\operatorname{Re}\lambda\geqslant\frac{1}{2}\alpha_\gamma^*$. It follows that if we take c_0 further small such that $c_0C<1$ then for c and λ in the set specified above, the operator $\lambda I-\mathbb{M}$ is invertible and the inverse is continuous. Hence, the desired assertion follows. □

7. The proof of Theorem 1.1

Proof of Theorem 1.1. We first assume that $\gamma>\gamma_*$. By Lemma 5.2 we see that $\mathbb{F}'(U_s)$ is a sectorial operator in X with domain X_0. In what follows we prove that the norm of X_0 coincides the graph norm of $\mathbb{F}'(U_s)$. From Section 6 we see that $\mathbb{F}'(U_s)=\mathbb{T}^{-1}\mathbb{M}\mathbb{T}$. Clearly,

$$C\|U\|_X\leqslant\|\mathbb{T}U\|_X\leqslant C^{-1}\|U\|_X\quad\text{and}\quad C\|U\|_{X_0}\leqslant\|\mathbb{T}U\|_{X_0}\leqslant C^{-1}\|U\|_X \tag{7.1}$$

for some constants $C>0$. Thus the graph norm of $\mathbb{F}'(U_s)$ is equivalent to the graph norm of $\mathbb{M}$. Next, let

$$\mathbb{T}_0=\begin{pmatrix} I & \sigma_s'(1)\Pi_0\\ 0 & I\end{pmatrix}.$$

Then we have $\mathbb{M}=\mathbb{T}_0\mathbb{M}_0$. Clearly, all estimates in (7.1) still hold when $\mathbb{T}$ is replaced by $\mathbb{T}_0$. Hence the graph norm of $\mathbb{M}$ is equivalent to the graph norm of $\mathbb{M}_0$. Clearly, as an unbounded linear operator in $W^{m-3,q}(\mathbb{B}^n)$ with domain $W^{m-1,q}(\mathbb{B}^n)$, the graph norm of $\mathcal{A}_0$ is equivalent to the norm of $W^{m-1,q}(\mathbb{B}^n)$. Also, we know that as an unbounded linear operator in $B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$ with domain $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$, the graph norm of $\mathcal{B}_\gamma$ is equivalent to the norm of $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ (cf. [16]). Besides, it is easy to see that $\mathcal{J}$ maps $W^{m-3,q}(\mathbb{B}^n)$ continuously into $B_{qq}^{m-2-1/q}(\mathbb{S}^{n-1})$, so that it is a compact operator from $W^{m-3,q}(\mathbb{B}^n)$ to $B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$. From these facts, we can easily show that the graph norm of $\mathbb{M}_0$ is equivalent to the norm of X_0. Hence, the graph norm of $\mathbb{F}'(U_s)$ is equivalent to the norm of X_0. This verifies that $\mathbb{F}'(U_s)$ satisfies the condition (B_1). By the results of Section 4 we see that $\mathbb{F}'(U_s)$ also satisfies the condition (B_2). Next we consider the condition (B_3). We first prove that $\mathbb{M}$ satisfies this condition. To this end we denote by $\mathbb{H}_1(\mathbb{S}^{n-1})$ the linear space of all first-order spherical harmonics, and for every integer k we introduce

$$\hat{B}_{qq}^{k-1/q}(\mathbb{S}^{n-1})=\left\{\rho\in B_{qq}^{k-1/q}(\mathbb{S}^{n-1})\colon\ \rho\text{ is orthogonal to }\mathbb{H}_1(\mathbb{S}^{n-1})\text{ in }L^2(\mathbb{S}^{n-1})\right\}.$$

We also denote $\hat{B}_{qq}^{\infty}(\mathbb{S}^{n-1})=\bigcap_{k=1}^{\infty}\hat{B}_{qq}^{k-1/q}(\mathbb{S}^{n-1})$. It can be easily shown that $\hat{B}_{qq}^{k-1/q}(\mathbb{S}^{n-1})$ is a closed subspace of $B_{qq}^{k-1/q}(\mathbb{S}^{n-1})$, and

$$B_{qq}^{k-1/q}(\mathbb{S}^{n-1})=\hat{B}_{qq}^{k-1/q}(\mathbb{S}^{n-1})\oplus\mathbb{H}_1(\mathbf{S}^{n-1})$$

(for any integer k). By (6.1) we see that $\ker(\mathcal{B}_\gamma)=\mathbb{H}_1(\mathbf{S}^{n-1})$. We denote $\hat{\mathcal{B}}_\gamma=\mathcal{B}_\gamma|_{\hat{B}_{qq}^{m-1/q}(\mathbb{S}^{n-1})}$, and split $\mathcal{J}$ into $\mathcal{J}_1+\mathcal{J}_2$ such that $\mathcal{J}_1v\in\hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$ and $\mathcal{J}_2v\in\mathbb{H}_1(\mathbb{S}^{n-1})$ for any $v\in W^{m-1,q}(\mathbb{B}^n)\cap$

$W_0^{1,q}(\mathbb{B}^n)$. We correspondingly split X_0 and X into $(W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)) \times \hat{B}_{qq}^{m-1/q}(\mathbb{S}^{n-1}) \times \mathbb{H}_1(\mathbb{S}^{n-1})$ and $W^{m-3,q}(\mathbb{B}^n) \times \hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1}) \times \mathbb{H}_1(\mathbb{S}^{n-1})$, respectively. Then

$$\mathbb{M} = \begin{pmatrix} c^{-1}\mathcal{A}_0 + \sigma_s'(1)\Pi_0(\mathcal{J}_1 + \mathcal{J}_2) & \sigma_s'(1)\Pi_0\hat{\mathcal{B}}_\gamma & 0 \\ \mathcal{J}_1 & \hat{\mathcal{B}}_\gamma & 0 \\ \mathcal{J}_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \hat{\mathbb{M}} & 0 \\ \hat{\mathcal{J}} & 0 \end{pmatrix},$$

where

$$\begin{aligned} \hat{\mathbb{M}} &= \begin{pmatrix} c^{-1}\mathcal{A}_0 + \sigma_s'(1)\Pi_0(\mathcal{J}_1 + \mathcal{J}_2) & \sigma_s'(1)\Pi_0\hat{\mathcal{B}}_\gamma \\ \mathcal{J}_1 & \hat{\mathcal{B}}_\gamma \end{pmatrix} \\ &= \begin{pmatrix} I & \sigma_s'(1)\Pi_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} c^{-1}\mathcal{A}_0 + \sigma_s'(1)\Pi_0\mathcal{J}_2 & 0 \\ \mathcal{J}_1 & \hat{\mathcal{B}}_\gamma \end{pmatrix} \equiv \hat{\mathbb{T}}_0\hat{\mathbb{M}}_1 \end{aligned}$$

and $\hat{\mathcal{J}} = (\mathcal{J}_2\, 0)$. We claim that $\hat{\mathcal{B}}_\gamma$ is an isomorphism from $\hat{B}_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ to $\hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$. Indeed, from (6.1) and the fact that $\mathcal{B}_\gamma$ maps $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ to $B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$ boundedly it is clear that $\hat{\mathcal{B}}_\gamma$ maps $\hat{B}_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ to $\hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$ boundedly and is an injection. Next, from (6.1) we see immediately that for any $\zeta \in \hat{B}_{qq}^{\infty}(\mathbb{S}^{n-1})$ there exists a unique $\eta \in \hat{B}_{qq}^{\infty}(\mathbb{S}^{n-1})$ such that $\mathcal{B}_\gamma\eta = \zeta$. Now assume that $\zeta \in \hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$. Let $\zeta_j \in \hat{B}_{qq}^{\infty}(\mathbb{S}^{n-1})$ $(j = 1, 2, \ldots)$ be such that $\zeta_j \to \zeta$ in $\hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$, and let $\eta_j \in \hat{B}_{qq}^{\infty}(\mathbb{S}^{n-1})$ be the solution of the equation $\mathcal{B}_\gamma\eta_j = \zeta_j$ $(j = 1, 2, \ldots)$. Take a real number s such that $s < m - 3 - 1/q - (n-1)(\frac{1}{2} - \frac{1}{q})$. Then $B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1}) \hookrightarrow H^s(\mathbb{S}^{n-1})$, where $H^s(\mathbb{S}^{n-1})$ stands for the usual Sobolev space. Thus $\zeta_j \to \zeta$ in $H^s(\mathbb{S}^{n-1})$. By (6.1) and the fact that $\alpha_{k,\gamma} \sim Ck^3$ we easily deduce that $\{\eta_j\}$ is a Cauchy sequence in $H^{s+3}(\mathbb{S}^{n-1})$. Let $\eta \in H^{s+3}(\mathbb{S}^{n-1})$ be the limit of $\{\eta_j\}$. By a standard argument we have

$$\|\rho\|_{B_{qq}^{m-1/q}(\mathbb{S}^{n-1})} \leqslant C\big(\|\rho\|_{H^{s+3}(\mathbb{S}^{n-1})} + \|\mathcal{B}_\gamma\rho\|_{B_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})}\big).$$

Applying this estimate to $\rho = \eta_j - \eta$, we conclude that $\eta_j \to \eta$ in $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$. Since $\hat{B}_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ is closed in $B_{qq}^{m-1/q}(\mathbb{S}^{n-1})$, we get $\eta \in \hat{B}_{qq}^{m-1/q}(\mathbb{S}^{n-1})$. This shows that $\hat{\mathcal{B}}_\gamma$ is a surjection. Hence, by the Banach inverse mapping theorem we see that $\hat{\mathcal{B}}_\gamma$ is an isomorphism from $\hat{B}_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ to $\hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$, as desired. Next, since $\mathcal{A}_0$ is an isomorphism from $W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)$ to $W^{m-3,q}(\mathbb{B}^n)$ and clearly $\sigma_s'(1)\Pi_0\mathcal{J}_2$ is a bounded operator from $W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)$ to $W^{m-3,q}(\mathbb{B}^n)$ (actually a compact operator), it follows that for c sufficiently small, $c^{-1}\mathcal{A}_0 + \sigma_s'(1)\Pi_0\mathcal{J}_2$ is an isomorphism from $W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)$ to $W^{m-3,q}(\mathbb{B}^n)$. By these results combined with the fact that $\mathcal{J}_1$ is a bounded operator from $W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)$ to $\hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$ (actually a compact operator), we immediately deduce that $\hat{\mathbb{M}}_1$ is an isomorphism from $(W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)) \times \hat{B}_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ to $W^{m-3,q}(\mathbb{B}^n) \times \hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$. Since $\hat{\mathbb{T}}_0$ is clearly a self-isomorphism on $W^{m-3,q}(\mathbb{B}^n) \times \hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$, we conclude that $\hat{\mathbb{M}}$ is an isomorphism from $(W^{m-1,q}(\mathbb{B}^n) \cap W_0^{1,q}(\mathbb{B}^n)) \times \hat{B}_{qq}^{m-1/q}(\mathbb{S}^{n-1})$ to $W^{m-3,q}(\mathbb{B}^n) \times \hat{B}_{qq}^{m-3-1/q}(\mathbb{S}^{n-1})$. This easily implies that $\mathbb{M}$ satisfies the condition (B_3). Now, since $\mathbb{F}'(U_s) = \mathbb{T}^{-1}\mathbb{M}\mathbb{T}$, it follows immediately that $\mathbb{F}'(U_s)$ also satisfies the condition (B_3). Finally, by Corollary 6.2 and Lemma 6.6 we see that

$$\omega_- = -\sup\big\{\operatorname{Re}\lambda:\ \lambda \in \sigma\big(\mathbb{F}'(U_s)\big) \setminus \{0\}\big\} > 0,$$

so that the condition (B_4) is also satisfied by $\mathbb{F}'(U_s)$. Hence, by Theorem 2.1 we get the assertion (i) of Theorem 1.1.

Next we assume that $0 < \gamma < \gamma_*$. Then there exists $k_0 \geqslant 2$ such that $\alpha_{k_0,\gamma} > 0$. By Lemma 6.5 and Corollary 6.2, this implies that for sufficiently small c, $\mathbb{F}'(U_s)$ has a positive eigenvalue. Furthermore, if $\alpha_{k_1,\gamma}, \alpha_{k_2,\gamma}, \dots, \alpha_{k_N,\gamma}$ are the all positive eigenvalues of $\mathcal{B}_\gamma$, then by Lemma 6.5 and a similar argument as in the proof of Lemma 6.6 we see that for c sufficiently small, $\lambda_{k_j,\gamma} = \alpha_{k_j,\gamma} + c\mu_{k_j,\gamma}(c)$ ($j = 1, 2, \dots, N$) are the all positive eigenvalues of $\mathbb{F}'(U_s)$, and the following estimate holds:

$$\sup\bigl\{\operatorname{Re}\lambda\colon\ \lambda \in \sigma(\mathbb{M}) \setminus \{0, \lambda_{k_1,\gamma}, \lambda_{k_2,\gamma}, \dots, \lambda_{k_N,\gamma}\}\bigr\} \leqslant \frac{1}{2}\max\bigl\{\alpha_k\colon\ k \geqslant 2,\ k \neq k_1, k_2, \dots, k_N\bigr\} < 0.$$

Thus by using Theorem 9.1.3 of [29], we obtain the assertion (ii) of Theorem 1.1. This completes the proof of Theorem 1.1. □

Added to the proofs

The proof of Lemma 6.6 neglected the fact that $0 \in \sigma(\mathcal{B}_\gamma)$, so that it is not rigorous. To remedy this deficiency we only need to shift the proof to the operator $\overline{\mathbb{M}} : X_0/\operatorname{Ker}\mathbb{M} \to X/\operatorname{Ker}\mathbb{M}$ induced by $\mathbb{M} : X_0 \to X$, and notice the facts that $\operatorname{Ker}\mathbb{M} = \{0\} \times \operatorname{Ker}\mathcal{B}_\gamma$ (by the proof of Corollary 6.4) and $\sigma(\overline{\mathbb{M}}) = \sigma(\mathbb{M}) \setminus \{0\}$.

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